Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-005-9010-x

Some Considerations on the Derivation of the Nonlinear Quantum Boltzmann Equation II: The Low Density Regime

D. Benedetto,[1] F. Castella,[2] R. Esposito,[3] and M. Pulvirenti[1]

Received 3 June 2005; accepted 11 November 2005
Published Online: February 16, 2006

In this paper we analyse the asymptotic dynamics of a system of N identical quantum particles in a *low-density* regime. Our approach follows the strategy introduced by the authors in a previous work,(2) to treat the simpler *weak coupling* regime. The time evolution of the Wigner transform of the one-particle reduced density matrix is represented by means of a perturbative series. The expansion is obtained upon iterating the Duhamel formula, in the spirit of the paper by Lanford.(32) For short times and small interaction potential, we rigorously prove that a *subseries* of the complete perturbative series, converges to the solution of the nonlinear Boltzmann equation that is physically relevant in the context. An important point is that we completely identify the cross-section entering the limiting Boltzmann equation, as being the Born series expansion of quantum scattering.

As in ref. 2, our convergence result is only partial, in that we merely characterize the asymptotic behaviour of a *subseries* of the complete original perturbative expansion. We only have plausibility arguments in the direction of proving that the terms we neglect, when going from the original series to its associated subseries, are indeed vanishing in the limit.

The present study holds in any dimension $d \geq 3$.

1. INTRODUCTION

A large quantum particle system in a rarefaction regime should be described by a Boltzmann equation. However, while the rigorous validity of the Boltzmann

[1] Dipartimento di Matematica, Università di Roma 'La Sapienza' P.le Aldo Moro, 5, 00185 Roma, Italy.

[2] IRMAR, Université de Rennes 1, Campus de Beaulieu, 35042 Rennes, Cedex, France; e-mail: francois.castella@univ-rennes1.fr

[3] Diparmimento di Matematica pura ed applicata, Università di L'Aquila, Coppito, 67100 L'Aquila, Italy.

equation has been proved for classical systems for short times,[32] or globally in time for special situations[27] (see ref. 10 for further comments), there is no rigorous analysis for the equivalent quantum systems.

The problem is physically relevant because quantum effects, although usually negligible at ordinary temperatures (except for few light molecules), happen to play a role in the applications at mesoscopic level. We refer, for example, to the treatment of electron gases in semiconductors. Physical references may be found in the textbooks,[1,12,40] as well as,[4,13] together with the articles[22] or[14]. We also quote[34] for a mathematically oriented presentation. Establishing a well founded quantum kinetic theory is certainly interesting not only from a conceptual viewpoint but also from a practical one. In fact, kinetic descriptions for quantum systems, beside dilute gases, include dense weakly interacting systems, as e.g. the electron gas in semiconductors, whose classical analogues rather yield diffusion processes.

One pragmatic way to introduce the Quantum Boltzmann equation (see e.g. ref. 9) is to solve the scattering problem in Quantum Mechanics and then to replace, in the classical Boltzmann equation, the classical cross section with the quantum one.

A better logically founded approach is to derive an evolution equation for the Wigner transform of a quantum state associated to a dilute particle system. Working on this equation, one can hope to recover, at the quantum level, the same physical arguments than those used at the classical level to obtain propagation of chaos and a suitable kinetic description for the one particle distribution function. We refer to the textbook[10] or the article[32] for the analysis of the classical case.

This is the approach we adopted in a previous paper,[2] to treat quantum N-particle systems *in the weak coupling regime*. In the present companion article, we adopt the similar point of view in order to investigate the more difficult *low-density regime*. One new, important, difficulty, lies in the identification of the limiting cross-section, as being the Born series expansion of quantum scattering. The latter is the natural transition rate to be recovered in the low-density situation. As a matter of fact, our asymptotic analysis gives a quite complicated expression for the limiting cross-section at first. The latter needs to be resummed in order to recognize the usual Born expansion. This resummation process is performed using a previous identification of the Born series, that was derived in the independent paper[7] in a quite different context, namely that of the *linear* Boltzmann equation in a low-density regime.

Let us come to some bibliographical comments. At the physical level, the question of passing from the Schrödinger equations to (linear or non-linear) Boltzmann equations is an old problem. We may quote,[29–31,38,45–48] as well as the textbooks quoted before. On the other hand, there is a large mathematical literature devoted to the asymptotic study of classical or quantum systems, in the low-density or weak-coupling regimes. Obviously these regimes can be considered either for a test particle moving in a random distribution of obstacles, or for systems of N coupled particles. As is well-known, a linear Boltzmann equation (or possibly a diffusive equation) is to be obtained in the first case, while a non-linear Boltzmann equation is expected in the second case. As a general reference on these questions, we wish to quote the book[(42)] as well as the review paper.[(43)] The situation of a classical test particle evolving in a random environment is studied in refs. 5 and 23 (low-density regime), and[(16)] (weak-coupling regime–here the linear Landau equation is obtained). These works provide linear transport equations in the limit. When a quantum test particle evolving in a random environment is considered, a linear Boltzmann equation can be derived in the weak coupling regime, either for short times (see refs. 25, 33 and 41), or globally in time (see refs. 19 and 20, see also ref. 11). The one-dimensional case is somehow pathological, see ref. 21. In a similar direction, the problem of wave motion in a random medium is also of interest for the applications (see ref. 28), and we also wish to mention[(37)] for the analysis of a weak coupling regime when the obstacles are random *in time* (and the underlying process is at once almost Markovian). The low-density regime for a quantum test particle is tackled in ref. 15 (case of an atom coupled to a gas), and more recently in ref. 17 (quantum particle in a random environment). The natural scattering cross-section is obtained in these two works, and ref. 17 actually recovers it under the form of the Born series expansion. In a similar spirit, though with quite different techniques, we wish to quote the computation by Nier[(36)] (see also ref. 35), which gives a powerful technique to recover the right scattering matrix (in the case of only *one* collision though). Last, the nonlinear situation (i.e. N particles systems) was studied in the major paper[(32)] (classical situation), which gives a complete analysis. In the quantum context, only partial results are available (refs. 24, 26 and more recently refs. 2, 3 and 18). The present work follows the route introduced in ref. 2, to tackle the low-density limit. Note that quantum N particles systems behave in a different way, depending whether particles are Fermions or Bosons, or simply are uncorrelated. In the weak-coupling regime, the limiting Boltzmann equation is quadratic for uncorrelated particles, while it involves a crucial corrective cubic term when particles obey the Fermi-Dirac or Bose-Einstein statistics.[(3,18)] In the low-density regime the effect of statistics is expected to vanish asymptotically, and the limiting Boltzmann equation formally is the same in all cases. This is the reason why the present text deals at once with uncorrelated particles.

Let us summarize the present contribution. This paper deals with a quantum N-particle system. The number N of particles goes to infinity along a low-density scaling. In this asymptotics, we represent the time evolution of the one-particle Wigner function in terms of a perturbative series expansion. On the basis of heuristic arguments already developed in the paper,[(2)] we neglect some terms and consider only a *subseries*. The latter is rigorously proved to converge, for short times and small interaction potential, to the solution of the Boltzmann equation with the suitable cross section, namely, the Born series expansion of quantum scattering. The present analysis is not a *complete* derivation of the Quantum Boltzmann equation, but only a *partial* one. However, we hope it constitutes a step in this direction.

Although we work in dimension 3, the present statements are easily extended in any space dimension $d \geq 3$. This is roughly due to the fact that the Schrödinger propagator $e^{it\Delta_x}$ ($x \in \mathbb{R}^d$) decays like $t^{-d/2}$ at infinity in time t, an integrable function of t whenever $d \geq 3$. Our analysis heavily relies on stationary phase computations, as well as appropriate representations of the various solutions of the hierarchies we need to handle.

We now leave further comments to the next sections, and come to establishing the model, the scaling, and the limiting Boltzmann equation. Our main Theorem is stated at the end of Sec. 2 below, and proved in the next sections. The main intermediate results of the analysis are Lemma 1, Theorem 2, Theorem 3, Theorem 4, Theorem 5, Theorem 6.

2. THE MODEL AND ITS SCALING LIMIT—MAIN RESULT

2.1. The Schrödinger Equation in the Low-Density Regime, and the Nonlinear Boltzmann Equation

We consider a N-particle quantum system in $\mathbb{R}^3$. We assume the mass of the particles, as well as $\hbar$, are normalized to one. The interaction is described by a single two-body potential ϕ, so that the total potential energy is:

$$U(x_1 \dots x_N) = \sum_{i<j} \phi(x_i - x_j).$$

The Schrödinger equation reads, in unscaled variables,

$$i\partial_t \Psi(t, X_N) = -\frac{1}{2}\Delta_{X_N}\Psi(t, X_N) + U(X_N)\Psi(t, X_N).$$

where $\Delta_{X_N} = \sum_{i=1}^N \Delta_{x_i}$, Δ_{x_i} is the Laplacian with respect to the x_i variable ($x_i \in \mathbb{R}^3$), and X_N is a shorthand notation for $X_N = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$.

We rescale the equation according to the hyperbolic space-time scaling

$$x \to \varepsilon x\ , \qquad t \to \varepsilon t. \tag{2.1}$$

In other words, we look at the behaviour of the N particles over long times of the order $O(1/\varepsilon)$, as well as large distances of the order $O(1/\varepsilon)$ (the speed of propagation of the particles is $O(1)$). Meanwhile, we leave the potential ϕ unchanged. It is an $O(1)$ perturbative potential, that acts over distances of the order $O(1)$ in the original variables. With this scaling, the original Schrödinger equation becomes

$$i\varepsilon\partial_t \Psi^\varepsilon(t, X_N) = -\frac{\varepsilon^2}{2}\Delta_{X_N}\Psi^\varepsilon(t, X_N) + U_\varepsilon(X_N)\Psi^\varepsilon(t, X_N, t),$$

where

$$U_\varepsilon(x_1 \dots x_N) = \sum_{i<j} \phi_\varepsilon(x_i - x_j), \qquad \text{and} \quad \phi_\varepsilon = \phi\left(\frac{x}{\varepsilon}\right). \tag{2.2}$$

The value at time t of the N particle wave function $\Psi^\varepsilon(t, X_N)$, is completely determined by Eq. (2.2) and the initial datum $\Psi^\varepsilon(0, X_N)$, left unspecified for the moment. Roughly speaking, we shall be interested in initial states that correspond to fully decorrelated particles. The quantitative assumptions on $\Psi^\varepsilon(0, X_N)$ are given later, see (2.15) below.

We want to analyze the limit $\varepsilon \to 0$ in the scaled Schrödinger equation (2.2), while keeping

$$N = \varepsilon^{-2}. \tag{2.3}$$

This kind of limit is usually called "low-density limit." In the classical context this is nothing but the Boltzmann-Grad limit (see e.g. ref. 10). This scaling makes sure that the typical distance between two particles is of the order $O(1/\varepsilon^{1/3})$. Hence the "collision" of two particles is typically a *rare* event, happening essentially *once* per unit time. Each collision deviates the particles by an $O(1)$ quantity, because the interaction potential is $O(1)$ in this regime. Another possible scaling is the *weak-coupling* limit, already studied in the companion paper.(2) This is, to some extent, a technically easier situation. In this case ϕ is scaled by a factor $\sqrt{\varepsilon}$, i.e. $\phi \to \sqrt{\varepsilon}\phi$, but N is larger, i.e. $N = O(\varepsilon^{-3})$. In other words, "collisions" are much more frequent and happen $O(1/\varepsilon)$ per unit time, but each collision has a small effect, of the order $O(\sqrt{\varepsilon})^2 = O(\varepsilon)$ (this is the Fermi Golden Rule, see below). Hence the cumulated effect of all the collisions is $O(1) = O(1/\varepsilon) \times O(\varepsilon)$ in this second regime.

Physically speaking, the situation is as follows. The evolution of the system is *a priori* described by the full N particle Schrödinger equation (2.2), an equation that strongly couples all particles. However, in the low-density regime, it is expected that the system has the following simpler asymptotic behaviour. First, it should tend to a system of N *decoupled* particles. Second, each particle should be well-described by a *one particle* density function $F(t, x, v)$, a *classical* object that relates the probability of finding the particle at position x with momentum v at time t. This effect is linked to the micro-macro space-time rescaling (2.1), that formally makes (2.2) a semi-classical Schrödinger equation. Finally, the function $F(t, x, v)$ should satisfy the nonlinear Boltzmann equation in the limit, namely,

$$\partial_t F(t,x,v) + v \cdot \nabla_x F = Q(F,F)(t,x,v),$$

where

$$\begin{aligned} Q(F,F) := 2\pi \int_{\mathbb{R}^9} & dv_* \, dv' \, dv'_* \; \delta(v + v_* - v' - v'_*) \; \delta \\ & \times \left(\left[\frac{v - v_*}{2} \right]^2 - \left[\frac{v' - v'_*}{2} \right]^2 \right) \\ & \times \Sigma^{\mathrm{low}} \left(\frac{v - v_*}{2} ; \frac{v' - v'_*}{2} \right) \left[F(t,x,v') \, F(t,x,v'_*) \right. \\ & \left. - F(t,x,v) \, F(t,x,v_*) \right]. \end{aligned} \tag{2.4}$$

Here, and as usual, v, v_* denote the two *ingoing* momenta, while v', v'_* are the two *outgoing* momenta.

The interpretation of (2.4) is standard. After the low-density asymptotics, the N particles system becomes decoupled, and the only "trace" of the initial coupling is eventually encoded in the so-called collision term $Q(F,F)$ in (2.4). The latter describes the encounter of two particles (x, v) and (x, v_*), sitting at the same position x, that get new momenta (x, v') and (x, v'_*) just after the collision. The collision process preserves total momentum, hence the $\delta(v + v_* - v' - v'_*)$. It also preserves kinetic energy, hence the $\delta([(v - v_*)/2]^2 - [(v' - v'_*)/2]^2) = \delta((v^2 + v_*^2 - v'^2 - v_*'^2)/2)$. Note that, when written in terms of absolute momenta v, v_*, etc., the kinetic energy is $v^2/2$, $v_*^2/2$, etc., i.e. mass is unity, while in terms of relative momenta $(v - v_*)/2$ and $(v' - v'_*)/2$, i.e. in the reference frame of the center of mass, kinetic energy becomes $[(v - v_*)/2]^2$ and $[(v' - v'_*)/2]^2$ (prefactor $1/2$ disappears). This is due to the fact that the relevant mass, in the reference frame of the center of mass, is the *reduced* mass $m_* = 1/2$. Last, the collision process has a specific cross-section denoted by $\Sigma^{\mathrm{low}}((v - v_*)/2; (v' - v'_*)/2)$ in (2.4), which is the probability that the initial relative momentum $(v - v_*)/2$ (momentum relative to the center of mass) takes the new value $(v' - v'_*)/2$ after the collision.

This is the only point where quantum mechanics enters the asymptotic description (2.4), and Eq. (2.4) is otherwise a perfectly classical object. In fact, the cross-section Σ^{low} does retain the microscopic features of the elementary collision, and it is to be computed along the quantum rules. More precisely, it is physically expected that Σ^{low} is given by:

$$\Sigma^{\mathrm{low}}(n,k) = |T(k,n)|^2, \tag{2.5}$$

where T is the standard T-matrix of quantum scattering associated with the potential ϕ, when written in the impulse representation. We recall in Sec. 3 the precise definition of this object and state some of its properties (see also ref. 39). Let us simply say here that Σ^{low} admits a complete series expansion in powers of the potential ϕ, which we write

$$\Sigma^{\text{low}}(n,k) = \Sigma_1(n,k) + \Sigma_2(n,k) + \cdots + \Sigma_m(n,k) + \cdots, \tag{2.6}$$

and, *by convention*, each Σ_m is homogeneous of degree $m+1$ in the potential. This is the so-called Born series expansion of quantum scattering. On top of that, we have the well-known formula for the lower order term:

$$\Sigma_1(n,k) = |\widehat{\phi}(n-k)|^2, \tag{2.7}$$

and $\widehat{\phi}$ is the Fourier transform of ϕ. This is the so-called Fermi Golden Rule.

In the weak-coupling regime, all the above statements hold true without modification, up to the fact that the cross-section Σ^{low} entering the limiting kinetic Eq. (2.4) then becomes Σ^{weak}, where

$$\Sigma^{\text{weak}}(n,k) = |\widehat{\phi}(n-k)|^2 = \Sigma_1(n,k). \tag{2.8}$$

In other words, the cross-section Σ^{weak} entering the weak coupling case is the first order approximation (in ϕ) of the cross-section Σ^{low} associated with the low-density. This explains why the latter case is technically more difficult than the former.

The next section is devoted to giving a more mathematical description in which sense Eq. (2.2) "converges" towards the Boltzmann Eq. (2.4) as $\varepsilon \to 0$. To summarize, let us simply say that the previous paper[2] gives a partial convergence result in the weak-coupling regime, and formula (2.8) is proved there, while the present text proposes a similar, partial convergence result in the low-density regime, and we do prove that the limiting cross-section is indeed given by (2.5). The general strategy of proof we adopt here, based on the use of the Wigner transform and a "quantum" BBGKY hierarchy, is borrowed from ref. 2. On the other hand, the identification of the cross-section uses[7] (see (2.5)).

Remark (normalisation). Throughout this article, the notation $\widehat{f}$ stands for the Fourier transform of f, normalized as follows:

$$\widehat{f}(h) = (\mathcal{F}_x f)(h) = \int_{\mathbb{R}^3} dx\, e^{-ih\cdot x} f(x),$$

$$f(x) = \left(\mathcal{F}_h^{-1} \hat{f}\right)(x) = \int_{\mathbb{R}^3} \frac{dh}{(2\pi)^3}\, e^{+ih\cdot x} \hat{f}(h).$$

2.2. The Wigner Transform, the "Quantum" BBGKY Hierarchy, and the Boltzmann Hierarchy

In order to quantitatively analyse the limit $\varepsilon \to 0$ in (2.2) along the low-density asymptotics, we introduce the Wigner function:

$$W^N(t, X_N, V_N) = \int_{\mathbb{R}^{3N}} dY_N \, e^{-iY_N \cdot V_N} \Psi^\varepsilon \left(t, X_N + \frac{\varepsilon}{2} Y_N\right) \overline{\Psi}^\varepsilon \left(t, X_N - \frac{\varepsilon}{2} Y_N\right), \tag{2.9}$$

where Ψ^ε is as in (2.2). A standard computation yields the transport equation:

$$\partial_t W^N(t, X_N, V_N) + V_N \cdot \nabla_{X_N} W^N = \left(\tilde{T}_N^\varepsilon W^N\right)(t, X_N, V_N), \tag{2.10}$$

where $V_N \cdot \nabla_{X_N} = \sum_{i=1}^N v_i \cdot \nabla_{x_i}$, and $\partial_t + V_N \cdot \nabla_{X_N}$ is the usual free stream operator. Also, we have introduced:

$$\tilde{T}_N^\varepsilon W^N := \sum_{0<k<\ell\leq N} \tilde{T}_{k,\ell}^\varepsilon W^N,$$

where,

$$\left(\tilde{T}_{k,\ell}^\varepsilon W^N\right)(t, X_N, V_N) := -\frac{i}{\varepsilon} \sum_{\sigma=\pm 1} \sigma \int_{\mathbb{R}^3} \frac{dh}{(2\pi)^3} \, e^{i\frac{h\cdot(x_k - x_\ell)}{\varepsilon}} \hat{\phi}(h)$$
$$\times W^N\left(t, x_1, v_1, \ldots, x_k, v_k - \frac{\sigma h}{2}, \ldots, x_\ell v_\ell + \frac{\sigma h}{2}, \ldots, x_N, v_N\right). \tag{2.11}$$

The operator $\tilde{T}_{k,\ell}^\varepsilon$ describes the (quantum) "collision" of particle k with particle ℓ, and the total operator $\tilde{T}_N^\varepsilon$ takes all possible "collisions" into account. Note that $x_k \neq x_\ell$ in (2.11), i.e. "collisions" occur at distant places in the quantum case. This fact that is penalized by the high-frequency term $\varepsilon^{-1} \exp\left(i \, h \cdot (x_k - x_\ell)/\varepsilon\right)$.

In order to pass to the limit in the transport Eq (2.10), we next need to introduce the partial traces of the Wigner transform, defined according to the formula, valid for $j = 1, \ldots, N-1$:

$$f_j^N(t, X_j, V_j) = \int_{\mathbb{R}^{6(N-j)}} dx_{j+1} \cdots dx_N \, dv_{j+1} \cdots dv_N \; W^N$$
$$\times (t, X_j, x_{j+1}, \ldots, x_N; V_j, v_{j+1}, \ldots, v_N). \tag{2.12}$$

Obviously, we also set $f_N^N = W^N$. The function $f_j^N(t, X_j, V_j)$ is roughly the reduced density function (in phase space) describing the state, at time t, of the j-particules subsystem.

From now on we shall suppose that all the particles are *identical*. As a consequence, the objects which we have introduced (Ψ^ε, W^N, f_j^N) are all *symmetric* in the exchange of particles. In particular, we do neglect the correlations between particles that are due to the quantum, bosonic or fermionic, statistics (see Eq. (2.15) below), because their effect is very small in the low density regime. This is not the case in the weak coupling limit, and we address to Benedetto *et al.*[3] for further discussions on this aspect.

Proceeding then as in the derivation of the BBGKY hierarchy for classical systems (see ref. 10), we readily arrive at the following hierarchy of equations (for $1 \leq j \leq N$):

$$\partial_t f_j^N(t, X_j, V_j) + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k} f_j^N = \tilde{T}_j^\varepsilon f_j + \tilde{C}_{j+1}^\varepsilon f_{j+1}^N. \tag{2.13}$$

This is the hierarchy to be studied in the present work, which naturally plays the role of a "quantum" BBGKY hierarchy. Here, the collision operator $\tilde{C}_{j+1}^\varepsilon$ is defined as:

$$\tilde{C}_{j+1}^\varepsilon = \sum_{k=1}^{j} \tilde{C}_{k,j+1}^\varepsilon, \qquad \text{where,}$$

$$\begin{aligned}\left(\tilde{C}_{k,j+1}^\varepsilon f_{j+1}^N\right)(t, X_j, V_j) = -i\,\frac{N-j}{\varepsilon} \sum_{\sigma=\pm1} \sigma \int_{\mathbb{R}^3} \frac{dh}{(2\pi)^3} \\ \times \int_{\mathbb{R}^6} dx_{j+1} dv_{j+1}\, \mathrm{e}^{i\frac{h\cdot(x_k - x_{j+1})}{\varepsilon}}\, \hat{\phi}(h) \\ \times f_{j+1}^N\Big(t, x_1, v_1, \ldots, x_k, v_k \\ - \frac{\sigma h}{2}, \ldots, x_{j+1}, v_{j+1} + \frac{\sigma h}{2}\Big).\end{aligned} \tag{2.14}$$

The operator $\tilde{C}_{k,j+1}^\varepsilon$ describes the "collision" of particle k, belonging to the j-particle subsystem, with a generic particle outside the subsystem, conventionally denoted by the number $j+1$ (this numbering uses the fact that all the particles are identical). The total operator $\tilde{C}_{j+1}^\varepsilon$ takes into account all such collisions. As usual,[10] Eq. (2.14) shows that the dynamics of the j-particle subsystem is governed by three effects: the free-stream operator, the collisions "inside" the subsystem (the $\tilde{T}$ term), and the collisions with particles "outside" the subsystem (the $\tilde{C}$ term).

There remains to specify the kind of initial data we take for the scaled Schrödinger Eq. (2.2) or, equivalently, for the hierarchy (2.13). From now on, we fix the initial value $\{f_j^N(0)\}_{j=1}^N$ of the solution $\{f_j^N(t)\}_{j=1}^N$ and we assume for simplicity that $\{f_j^N(0)\}_{j=1}^N$ is factorized, that is, for all $j = 1, N$

$$f_j^N(0) = f_0^{\otimes j}, \tag{2.15}$$

where f_0 is a one-particle Wigner function which we assume to be a probability distribution. We remind that the quantum state, whose Wigner transform is a general positive f_0, is not in general a wave function but rather a density matrix. As a consequence the evolution equation we have to use is not the Schrödinger Eq. (2.2) but rather the Heisenberg equation for the density matrix. In both cases the corresponding Wigner equation is anyhow (2.10), hence we shall not comment further on this aspect.

In the present picture, the quantitative meaning of the "convergence" of the scaled Schrödinger Eq. (2.2) towards the Boltzmann Eq. (2.4) takes the following form. We expect that the one particle reduced Wigner transform $f_1^N(t)$ converges towards the solution $F(t)$ to the nonlinear Boltzmann Eq. (2.4), with initial datum $F(0) = f_0$. Even more, we expect that for any j, the j particle reduced Wigner transform $f_j^N(t)$ converges towards the j tensor product $F(t)^{\otimes j}$.

As a consequence, and in order to measure the convergence of $f_j^N(t)$ towards $F(t)^{\otimes j}$, we now define the j-tensor product

$$F_j(t, X_j, V_j) := F(t)^{\otimes j}(t, X_j, V_j),$$

where $F(t)$ satisfies the Boltzmann Eq. (2.4), with initial datum $F(0) = f_0$. It is readily observed that the sequence F_j satisfies the following hierarchy, analogous to (2.13), and called "Boltzmann hierarchy,"

$$\partial_t F_j + \sum_{k=1}^{j} v_k \cdot \nabla_{x_k} F_j = C_{j+1} F_{j+1}, \qquad F_j(0) = f_0^{\otimes j}. \tag{2.16}$$

Here we defined the classical collision operator (note that $x_k = x_{j+1}$ here):

$$C_{j+1} = \sum_{k=1}^{j} C_{k,j+1}, \qquad \text{where,}$$

$$\begin{aligned}(C_{k,j+1}F_j)(t, X_j, V_j) = 2\pi \int_{\mathbb{R}^9} & dv_{j+1}\, dv'_k\, dv'_{j+1}\; \delta(v_k + v_{j+1} - v'_k - v'_{j+1}) \\ & \times \delta\left(\frac{v_k^2}{2} + \frac{v_{j+1}^2}{2} - \frac{v_k^{'2}}{2} - \frac{v_{j+1}^{'2}}{2}\right) \\ & \times \Sigma^{\text{low}}(v_k - v_{j+1}; v'_k - v'_{j+1}) \\ & \times [F_{j+1}(t, x_1, v_1, \dots, x_k, v'_k, \dots, x_j, v_j, x_k, v'_{j+1}) \\ & - F_{j+1}(t, x_1, v_1, \dots, x_k, v_k, \dots, x_j, v_j, x_k, v_{j+1})].\end{aligned} \tag{2.17}$$

We are now in position to state our main result in the next section.

Remark (notation). Occasionally, and in accordance with the Born series expansion $\Sigma^{\mathrm{low}} = \sum_{m\geq 1} \Sigma_m$, of the cross-section (see (2.6)), we shall need to decompose the collision operator $C_{k,j+1}$ associated with Σ^{low} (see (2.17)), into

$$C_{k,j+1} = \sum_{m\geq 1} C^m_{k,j+1}, \tag{2.18}$$

up to defining the $(m+1)$-th order collision operator associated with Σ_m

$$\begin{aligned}\left(C^m_{k,j+1} F_j\right)(t, X_j, V_j) = 2\pi \int_{\mathbb{R}^9} & dv_{j+1}\, dv'_k\, dv'_{j+1}\; \delta(v_k + v_{j+1} - v'_k - v'_{j+1}) \\ & \times \delta\left(\left[\frac{v_k - v_{j+1}}{2}\right]^2 - \left[\frac{v'_k - v'_{j+1}}{2}\right]^2\right) \\ & \times \Sigma_m\left(\frac{v_k - v_{j+1}}{2}; \frac{v'_k - v'_{j+1}}{2}\right) \\ & \times [F_{j+1}(t, \ldots, x_k, v'_k, \ldots, x_j, v_j, x_k, v'_{j+1}) \\ & - F_{j+1}(t, \ldots, x_k, v_k, \ldots, x_j, v_j, x_k, v_{j+1})],\end{aligned} \tag{2.19}$$

i.e. the same formula than (2.17), but with the full cross-section Σ^{low} replaced by the $(m+1)$-th order contribution Σ_m.

2.3. Statement of Our Results

In order to compare both hierarchies (2.13) and (2.16), one can try to handle them as in the case of the Boltzmann-Grad limit for classical systems, namely one may study the asymptotic behaviour of the *explicit solutions* of (2.13) resp. (2.16), when expressed as complete series expansions obtained upon iterating the Duhamel formula.

In the case of the "quantum" BBGKY hierarchy, this iterative procedure gives

$$\begin{aligned} f^N_j(t) = S^\varepsilon_{\mathrm{int}}(t)\, f^N_j(0) + \sum_{n=1}^{N-j} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\ \times S^\varepsilon_{\mathrm{int}}(t - t_1)\, \tilde{C}^\varepsilon_{j+1}\, S^\varepsilon_{\mathrm{int}}(t_1 - t_2) \\ \times \tilde{C}^\varepsilon_{j+2} \cdots S^\varepsilon_{\mathrm{int}}(t_{n-1} - t_n)\, \tilde{C}^\varepsilon_{j+n}\, S^\varepsilon_{\mathrm{int}}(t_n)\, f^N_{j+n}(0). \end{aligned} \tag{2.20}$$

Here $S^\varepsilon_{\mathrm{int}}(t) f_j$ is the j-particle interacting flow, namely the solution to the initial value problem:

$$\left(\partial_t + V_j \cdot \nabla_{x_j}\right) S^\varepsilon_{\mathrm{int}}(t) f_j = \frac{1}{\varepsilon} T^\varepsilon_j\, S^\varepsilon_{\mathrm{int}}(t) f_j, \qquad S^\varepsilon_{\mathrm{int}}(0) f_j = f_j. \tag{2.21}$$

We may further expand $S^\varepsilon_{\rm int}(t)$ as a perturbation of the free flow $S(t)$, defined as

$$(S(t)f_j)(X_j, V_j) = f_j(X_j - V_j t, V_j). \tag{2.22}$$

This gives the following representation of the interacting flow $S^\varepsilon_{\rm int}$:

$$\begin{aligned} S^\varepsilon_{\rm int}(t) f_j = S(t) f_j + \sum_{m\geq 1} \frac{1}{\varepsilon^m} \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{m-1}} d\tau_m S(t-\tau_1)\, T_j^\varepsilon \\ \times S(\tau_1 - \tau_2)\, T_j^\varepsilon \cdots S(\tau_{m-1} - \tau_m)\, T_j^\varepsilon\, S(\tau_m)\, f_j(0). \end{aligned} \tag{2.23}$$

In order to shorten notations, throughout this article, we shall write (2.23) under the compact form

$$S^\varepsilon_{\rm int}\, f_j = S f_j + \sum_{m\geq 1} \underbrace{S\tilde{T}_j^\varepsilon \ldots S\tilde{T}_j^\varepsilon}_{m \text{ times}} S f_j(0) = S\, f_j + \sum_{m\geq 1} \left(S\, \tilde{T}_j^\varepsilon\right)^m\, S\, f_j(0). \tag{2.24}$$

This turns out to be a convenient abuse of notations. Using the shorthand writing (2.24), Eq. (2.23) and (2.20) become

$$\begin{aligned} f_j^N(t) &= S^\varepsilon_{\rm int}\, f_j^N(0) + \sum_{n=1}^{N-j} S^\varepsilon_{\rm int}\, \tilde{C}^\varepsilon_{j+1}\, S^\varepsilon_{\rm int}\, \tilde{C}^\varepsilon_{j+2} \cdots S^\varepsilon_{\rm int}\, \tilde{C}^\varepsilon_{j+n}\, S^\varepsilon_{\rm int}\, f^N_{j+n}(0), \\ S^\varepsilon_{\rm int} &= S + \sum_{m\geq 1} \left(S\tilde{T}_j^\varepsilon\right)^m S. \end{aligned}$$

Last, using the fact that the initial datum is factorized (2.15), and combining the above relations, we eventually obtain the *equality*

$$\begin{aligned} f_j^N(t) &= S(t) f_0^{\otimes j} \\ &\quad + \sum_{n=1}^{N-j} \left(\sum_{m_0\geq 0} \left(S\, \tilde{T}_j^\varepsilon\right)^{m_0} \right) \left(\sum_{m_1\geq 0} S\, \tilde{C}^\varepsilon_{j+1} \left(S\, \tilde{T}^\varepsilon_{j+1}\right)^{m_1} \right) \\ &\quad \times \left(\sum_{m_2\geq 0} S\, \tilde{C}^\varepsilon_{j+2} \left(S\, \tilde{T}^\varepsilon_{j+2}\right)^{m_2} \right) \cdots \\ &\quad \cdots \left(\sum_{m_n\geq 0} S\, \tilde{C}^\varepsilon_{j+n} \left(S\tilde{T}^\varepsilon_{j+n}\right)^{m_n} \right) S\, f_0^{\otimes j+n}. \end{aligned} \tag{2.25}$$

Formula (2.25) translates the fact that the dynamics of the j particle subsystem may be decomposed as follows. First the system alternates free flights with collisions *inside* the subsystem, this is the term $(S\,\tilde{T}_j^\varepsilon)^{m_0}$. Then one particle hits particle number $j+1$ *outside* the subsystem, this is the term $S\,\tilde{C}^\varepsilon_{j+1}$. Last, after the collision is done, the process starts again: free flights and collisions inside the $j+1$ particle subsystem (term $(S\,\tilde{T}^{\varepsilon}_{j+1})^{m_1}$), "creation" of a new, $j+2$, particle, etc.

Similarly, solving the Boltzmann hierarchy (2.16) as we did with (2.13), we recover

$$F_j(t) = S(t) f_0^{\otimes j} + \sum_{n=1}^{N-j} (S\,C_{j+1})(S\,C_{j+2}) \cdots (S\,C_{j+n})\, S\, f_0^{\otimes j+n}, \tag{2.26}$$

Now, formulae (2.25) and (2.26) give the *explicit* value of f_j^N resp. F_j. They are the natural starting point for an analysis in the spirit of the Lanford proof for the classical Boltzmann equation. The goal is, in this perspective, to derive uniform bounds on, and pass to the limit in, the series expansion (2.25), proving the convergence towards the analogous expansion (2.26).

As we explain in more detail in ref. 2, we are *not* able to analyse the full series expansion (2.25), that relates the actual value of $f_j^N(t)$. We are only able to treat a *subseries*, denoted $\tilde{f}_j^N(t)$ in the sequel, whose quantitative value is given in (2.29) below. Thus, the result presented in this paper has two aspects. On the one hand, we can *rigorously show* that $\tilde{f}_j^N(t)$ goes to $F_j(t) = F(t)^{\otimes j}$. This is the main result. On the other hand, we have already given in ref. 2 *heuristic arguments* indicating that the true series expansion (2.25) defining $f_j^N(t)$, and its associated subseries $\tilde{f}_j^N(t)$, should be asymptotic to one another. We do not recall here the arguments detailed in ref. 2. Let us simply say that, in essence, we know how to prove that the *lower order terms* that we skip when passing from the exact series expansion of $f_j^N(t)$, to the subseries $\tilde{f}_j^N(t)$, are indeed negligible. However, it seems difficult to develop a general strategy to treat at once all terms entering the expansion of $f_j^N(t)$, and even more difficult to prove enough a priori bounds to make the series defining $f_j^N(t)$ converge in any uniform sense.

Let us now define the subseries $\tilde{f}_j^N(t)$ of interest. In view of the expansion (2.25) of f_j^N, we first *claim* that all the relevant terms in (2.25) are those corresponding to

$$m_0 = 0, \text{ together with } m_1 \geq 1, m_2 \geq 1, \ldots, m_n \geq 1.$$

In other words, as $\varepsilon \to 0$, we claim that f_j^N is asymptotic to,

$$f_j^N(t) \sim S(t) f_j^N(0) + \sum_{n=0}^{N-j} \left(\sum_{m_1 \geq 1} S\,\tilde{C}^{\varepsilon}_{j+1} \left(S\,\tilde{T}^{\varepsilon}_{j+1}\right)^{m_1} \right) \times \left(\sum_{m_2 \geq 1} S\,\tilde{C}^{\varepsilon}_{j+2} \left(S\,\tilde{T}^{\varepsilon}_{j+2}\right)^{m_2} \right) \cdots\cdots \left(\sum_{m_n \geq 1} S\,\tilde{C}^{\varepsilon}_{j+n} \left(S\,\tilde{T}^{\varepsilon}_{j+n}\right)^{m_n} \right) S\, f_0^{\otimes j+n}. \tag{2.27}$$

In the heuristic picture given above, this means that the contribution of the first m_0 collisions *inside* the j particle subsystem is claimed negligible, unless $m_0 = 0$. Second, expanding each "collision" term in (2.27) into $\tilde{C}^\varepsilon_{j+1} = \sum_{r=1}^{j} \tilde{C}^\varepsilon_{r,j+1}$, and $\tilde{T}^\varepsilon_j = \sum_{r,\ell=1}^{j} \tilde{T}^\varepsilon_{r,\ell}$, we also *claim* that f_j^N is asymptotic to,

$$f_j^N(t) \sim S(t) f_j^N(0) + \sum_{n=0}^{N-j} \left(\sum_{r_1=1}^{j} \sum_{m_1 \geq 1} S\, \tilde{C}^\varepsilon_{r_1,j+1} \left(S\, \tilde{T}^\varepsilon_{r_1,j+1}\right)^{m_1} \right) \times \left(\sum_{r_2=1}^{j+1} \sum_{m_2 \geq 1} S\, \tilde{C}^\varepsilon_{r_2,j+2} \left(S\, \tilde{T}^\varepsilon_{r_2,j+2}\right)^{m_2} \right) \cdots \times \left(\sum_{r_n=1}^{j+n-1} \sum_{m_n \geq 1} S\, \tilde{C}^\varepsilon_{r_n,j+n} \left(S\, \tilde{T}^\varepsilon_{r_n,j+n}\right)^{m_n} \right) S\, f_0^{\otimes j+n}. \tag{2.28}$$

In other terms, we claim that the dynamics of the j-particle subsystem is only made up of *collision/recollision events*, in the asymptotics $\varepsilon \to 0$. When particle r_1 meets particle $j+1$, it immediately recollides with it, until the next particle $j+2$ is created, and so on.

Summarizing, we *define* the sequence $\{\tilde{f}_j^N\}_{j=1}^N$ by

$$\tilde{f}_j^N(t) = S(t) f_j^N(0) + \sum_{n=0}^{N-j} \left(\sum_{r_1=1}^{j} \sum_{m_1 \geq 1} S\, \tilde{C}^\varepsilon_{r_1,j+1} \left(S\, \tilde{T}^\varepsilon_{r_1,j+1}\right)^{m_1} \right) \times \left(\sum_{r_2=1}^{j+1} \sum_{m_2 \geq 1} S\, \tilde{C}^\varepsilon_{r_2,j+2} \left(S\, \tilde{T}^\varepsilon_{r_2,j+2}\right)^{m_2} \right) \cdots \times \left(\sum_{r_n=1}^{j+n-1} \sum_{m_n \geq 1} S\, \tilde{C}^\varepsilon_{r_n,j+n} \left(S\, \tilde{T}^\varepsilon_{r_n,j+n}\right)^{m_n} \right) S\, f_0^{\otimes j+n}. \tag{2.29}$$

The sequence $\tilde{f}_j^N$ relates the value of the right-hand-side in (2.28), it is obviously a *subseries* of the true series expansion (2.25) that defines f_j^N. We *claim* that the true series $f_j^N(t)$ and the modified series $\{\tilde{f}_j^N\}_{j=1}^N$ are asymptotic in the present regime:

$$f_j^N(t) \underset{\varepsilon \to 0}{\sim} \tilde{f}_j^N(t). \tag{2.30}$$

The main result of the present paper lies in the rigorous proof of convergence of the series expansion (2.29) of $\tilde{f}_j^N$ towards that of F_j (2.26). More precisely, we prove the

Main Theorem

Take a potential ϕ and an initial datum f_0 that are "smooth," in the sense that $N_1(f_0) < \infty$ and $N_0(\phi) < \infty$. The norms N_1 and N_0 are defined in (4.22), (4.23), and (4.25) below.

Assume the potential ϕ is "small," in the sense that $N_0(\phi) \leq c_1$, for some universal constant $c_1 > 0$.

Then, for short times $0 \leq t \leq c_2$ (where $c_2 > 0$ is some universal constant), the subseries $\tilde{f}_j^N$ goes, as $\varepsilon \to 0$, to $(F(t))^{\otimes j}$, where $F(t)$ solves the Boltzmann Eq. (2.4) with the cross-section Σ^{low} defined in (2.5), and initial data $F(0) = f_0$. The convergence holds in the topology of functions that are continuous in time, endowed with the norm N_1 in the variables X_j and V_j.

The remainder part of this paper is organised as follows: the preparatory Sec. 2.4 gives the main lines of our proof; Sec. 3 is devoted to recalling some formulae related with the Born series of quantum scattering to be used next; Secs. 4 to 8 gives the detailed proof of our main result.

2.4. Strategy of Proof

Before going to the proofs, we sketch here the general strategy adopted in this paper, and give some short-hand notations to be used later.

The expansion (2.29) asserts

$$\tilde{f}_j^N(t) = S(t) f_0^{\otimes j} + \sum_{n=1}^{N-j} \left[\sum_{r_1=1}^{j} \left(\sum_{m_1 \geq 1} S\, \tilde{C}^{\varepsilon}_{r_1, j+1} \left(S\, \tilde{T}^{\varepsilon}_{r_1, j+1} \right)^{m_1} \right) \right]$$
$$\times \left[\sum_{r_2=1}^{j+1} \left(\sum_{m_2 \geq 1} S\, \tilde{C}^{\varepsilon}_{r_2, j+2} \left(S\, \tilde{T}^{\varepsilon}_{r_2, j+2} \right)^{m_2} \right) \right] \cdots$$
$$\times \left[\sum_{r_n=1}^{j-n-1} \left(\sum_{m_n \geq 1} S\, \tilde{C}^{\varepsilon}_{r_n, j+n} \left(S\, \tilde{T}^{\varepsilon}_{r_n, j+n} \right)^{m_n} \right) \right] S\, f_0^{\otimes j+n}. \tag{2.31}$$

Let us define for later convenience, for any particle names a and b, the "collision operators":

$$C^{m,\varepsilon}_{a,b} = \tilde{C}^{\varepsilon}_{a,b} \,(S\, \tilde{T}^{\varepsilon}_{a,b})^m, \qquad \text{and} \qquad C^{\varepsilon}_{a,b} = \sum_{m \geq 1} C^{m,\varepsilon}_{a,b}. \tag{2.32}$$

With this short-hand writing, Eq. (2.31) takes the form

$$\tilde{f}_j^N(t) = S(t) f_0^{\otimes j} + \sum_{n=1}^{N-j} \left[\sum_{r_1=1}^{j} S\, C^{\varepsilon}_{r_1, j+1} \right]$$

$$\times \left[\sum_{r_2=1}^{j+1} S\, C^{\varepsilon}_{r_2,j+2} \right] \cdots \left[\sum_{r_n=1}^{j+n-1} S\, C^{\varepsilon}_{r_n,j+n} \right] S\, f_0^{\otimes j+n}, \qquad (2.33)$$

where each operator $S\, C^{\varepsilon}_{r_1,j+1}$, etc., decomposes into

$$S\, C^{\varepsilon}_{r_1,j+1} = \sum_{m\geq 1} S\, C^{m,\varepsilon}_{r_1,j+1}. \qquad (2.34)$$

On the other hand, the iterative resolution of the Boltzmann hierarchy for $F_j(t)$ readily led to (2.26), which reads, upon expanding $C_{j+1} = \sum_{r_1} C_{r_1,j+1}$ and so on,

$$F_j(t) = S(t) f_0^{\otimes j} + \sum_{n=1}^{N-j} \left[\sum_{r_1=1}^{j} S\, C_{r_1,j+1} \right]$$

$$\times \left[\sum_{r_2=1}^{j+1} S\, C_{r_2,j+2} \right] \cdots \left[\sum_{r_n=1}^{j+n-1} S\, C_{r_n,j+n} \right] S\, f_0^{\otimes j+n}, \qquad (2.35)$$

On the more, each operator $S\, C_{r_1,j+1}$, etc., decomposes into (see (2.18) and (2.19)),

$$S\, C_{r_1,j+1} = \sum_{m\geq 1} S\, C^{m}_{r_1,j+1}. \qquad (2.36)$$

In view of the parallel formulae (2.33) and (2.35), the proof that $\tilde{f}_j^N(t)$ goes to F_j boils down, in essence, to proving that, for any particle names a and b, the operator

$$S\, C^{\varepsilon}_{a,b} = \sum_{m\geq 1} S\, C^{m,\varepsilon}_{a,b}$$

goes to its classical counterpart

$$S\, C_{a,b} = \sum_{m\geq 1} S\, C^{m}_{a,b},$$

and we iterate the information. From a purely technical point of view, it turns out that unfortunately, the "iteration" step cannot be performed nicely. For this reason, we shall need to really handle the full series expansions (2.33) and (2.35). Hence our strategy of proof will be the following:

- As a preliminary step, we first recall in Sec. 3 below the necessary informations about the T matrix of quantum scattering that enters the cross-Sec. (2.5). In particular, we recall the non-trivial representation formula (3.8) for the Born series.

- Now, in view of the parallel decompositions (2.34) and (2.36), we completely analyse in Secs. 4 and 5 the asymptotic behaviour of the reference term

$$S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes 2}.$$

for each given values of the particle names a and b, and for a given index $m \geq 1$. Uniform bounds are given in Theorem 2. Theorem 3 then establishes the non trivial identity

$$S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes 2} \to_{\varepsilon\to 0} S\,C^{m}_{a,b}\;S\,f_0^{\otimes 2}.$$

As an immediate consequence, for any value of j, the similar results are deduced for the more general $S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes j}$.

This is really the key step of our proof.

Two important ingredients are a careful representation of the term $S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes 2}$ (formula (4.7)), together with a detailed analysis of the oscillatory integrals that define $S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes 2}$. The second ingredient ends with the explicit computation of a quadratic phase, that makes it possible to apply the stationnary phase theorem (Lemma 1). As a third and last tool, and in order to identify the limit and recognize the cross-Sec. (2.5) as being the Born series expansion of quantum scattering, we also need to use the representation formula (3.8) recalled in Sec. 3.

- Next, we come in section 5.2 to the computation of the sum (related with the Born series expansion $\Sigma^{\text{low}} = \sum_{m\geq 1}\Sigma_m$):

$$\sum_{m\geq 1} S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes 2} = S\,C^{\varepsilon}_{a,b}\;S\,f_0^{\otimes 2}.$$

Theorem 4 gives a uniform bound, and establishes that this term goes to its classical counterpart

$$\sum_{m\geq 1} S\,C^{m}_{a,b}\;S\,f_0^{\otimes 2} = S\,C_{a,b}\;S\,f_0^{\otimes 2}.$$

As an immediate consequence, this gives the behaviour of $\sum_{m\geq 1} S\,C^{m,\varepsilon}_{a,b}\;S\,f_0^{\otimes j} = S\,C^{\varepsilon}_{a,b}\;S\,f_0^{\otimes j}$, for any value of j. This step again uses the representation formula (3.8).

- The above two informations are not quite enough to pass to the limit in the true series expansion (2.33). This is the reason why we give in Sec. 7 the technical details that allow to pass to the limit in the case with "two iterations," namely in

$$S\,C^{\varepsilon}_{r_1,j+1} S\,C^{\varepsilon}_{r_2,j+2}\;S\,f_0^{\otimes j+2} = \left(\sum_{m_1} S\,C^{m_1,\varepsilon}_{r_1,j+1}\right)\left(\sum_{m_2} S\,C^{m_2,\varepsilon}_{r_2,j+2}\right)\;S\,f_0^{\otimes j+2}.$$

We prove a uniform bound and show that the limit is given by the classical counterpart

$$S\,C_{r_1,j+1} S\,C_{r_2,j+2}\, S\, f_0^{\otimes j+2} = \left(\sum_{m_1} S\,C^{m_1}_{r_1,j+1}\right)\left(\sum_{m_2} S\,C^{m_2}_{r_2,j+2}\right) S\, f_0^{\otimes j+2}.$$

This is more a technical step. The main result is Theorem 5.

- To be complete, we last explain in the conclusive Sec. 8 how to pass to the limit in the full series expansion (2.33), in particular how to perform the summation in the index n.

3. THE BORN SERIES EXPANSION, AND ITS IDENTIFICATION

As explained in Sec 2.1, the T matrix of quantum scattering plays a key role in this paper (see (2.5)). For this reason, we recall here some results on T, and the associated Born series expansion.

Definition. *The T-matrix of quantum scattering associated with the potential ϕ is defined, see e.g. ref. 39*[4] *as*[5]

$$T(n,k) = \int_{\mathbb{R}^3} \mathrm{e}^{-inx}\,\phi(x)\,\psi(x,k)\,dx, \tag{3.1}$$

where the distorted plane wave $\psi(x,k)$ is given by

$$\psi(x,k) = \mathrm{e}^{ikx} - \left[-\Delta_x - k^2 + i0^+\right]^{-1} \phi(x)\,\psi(x,k). \tag{3.2}$$

With this definition, it is standard to observe that the so-called Lippmann-Schwinger relation (3.2) may be solved in an iterative way $\psi = \mathrm{e}^{ikx} - [-\Delta_x - k^2 + i0^+]^{-1}\phi(x)\,\mathrm{e}^{ikx} - \cdots$. This makes it possible to express T as a power series expansion in the potential ϕ. Plugging such an expansion into $\Sigma^{\mathrm{low}}(n,k) = |T(k,n)|^2$ (see (2.5)), results in the so-called Born series expansion of quantum scattering, namely,

$$\Sigma^{\mathrm{low}}(n,k) = |T(k,n)|^2 \equiv \Sigma_1(n,k) + \Sigma_2(n,k) + \cdots + \Sigma_m(n,k) + \cdots, \tag{3.3}$$

[4] Note that the definition of T we use here differs from the standard on by a (harmless) factor $(2\pi)^3$.

[5] In fact, the T matrix is related with the scattering between the operators $-\Delta$ and $-\Delta + \phi$. Note the important fact that we definitely use here $H_0 = -\Delta$ as the reference operator in the scattering process, i.e. we take a mass set to $1/2$. This is due to the fact that the nonlinear Boltzmann equation we partially derive in this text involves a cross-section which is nicely written in the reference frame of the center of mass, hence the relevant mass is the reduced mass $m_* = 1/2$—see (2.4) and subsequent comments.

where, by convention, each Σ_m is homogeneous of degree $m+1$ in the potential ϕ. This defines Σ_m. The following property is easily established (see e.g. ref. 39).

Property. *Let Σ_m ($m \geq 1$) be as in (3.3), then $\Sigma_1(n,k) = |\hat{\phi}(n-k)|^2$ and, for $m \geq 2$, we have*

$$\begin{aligned}\Sigma_m(n,k) &= \frac{i^{m+1}}{(2\pi)^{m-1}} \\ &\times \sum_{s=0}^{m-1} (-1)^{s+1} \int_{\mathbb{R}^{3(m-1)}} dk_1 \ldots dk_{m-1} \Delta(k_1,n) \cdots \Delta(k_s,n) \Delta(n,k_{s+1}) \cdots \\ &\times \Delta(n,k_{m-1}) \hat{\phi}(n-k_1) \hat{\phi}(k_1-k_2) \cdots \hat{\phi}(k_s-k) \hat{\phi}(k-k_{s+1}) \hat{\phi}(k_{s+1}-k_{s+2}) \\ &\times \cdots \; \hat{\phi}(k_{m-1}-n). \end{aligned} \tag{3.4}$$

Here, we defined

$$\Delta(n,p) := \int_0^{+\infty} ds \;\; \exp(is[n^2-p^2]) \qquad \left(= \frac{i}{n^2-p^2+i0}\right), \tag{3.5}$$

an object which makes sense as an oscillatory integral, hence as a distribution. It also has the value

$$\Delta(n,p) = \pi \delta(n^2-p^2) + i \text{p.v.} \left(\frac{1}{n^2-p^2}\right).$$

The expansion (3.3) together with formula (3.4) give the "natural" value of the Born series expansion of quantum scattering. In ref. 17, Eng and Erdös study the behaviour of a quantum test particle in a random environment, in a low-density regime: their analysis gives a cross-section that is naturally expressed as the power series expansion (3.3)–(3.4). In a similar context, the analysis given in ref. 15 gives an even stronger result: here, the obtained cross-section is directly expressed in terms of the above T-matrix, and the author does not even need to further expand T in terms of the potential.

At variance, our approach gives a quite complicated value of the cross-section in a first step, and the identification with formulae (3.3)–(3.4) requires a second, independent, step. This is somehow a drawback of our approach : the cross-section does not naturally come in the form (3.3)–(3.4). This is the reason why we now introduce, for later purposes, the following linear Boltzmann operator. At first glance, it has nothing to do with the Born series expansion.

Definition. *Take f a smooth test function, and $m \geq 1$ be an integer. Define the quantity (linear Boltzmann operator),*

$$Q_m f(n) = 2 \text{ Re } \frac{(-i)^{m+1}}{(2\pi)^{3m}} \sum_{\varepsilon_1,\ldots,\varepsilon_m} (-1)^{\varepsilon_1+\cdots+\varepsilon_m}$$
$$\times \int_{\mathbb{R}^{3(m-1)}} dk_1 \ldots dk_m \, \Delta(n - \varepsilon_1 k_1, n + \tilde{\varepsilon}_1 k_1)$$
$$\times \cdots \Delta(n - \varepsilon_1 k_1 - \cdots - \varepsilon_m k_m, n + \tilde{\varepsilon}_1 k_1 + \cdots + \tilde{\varepsilon}_m k_m) \, \hat{\phi}(k_1)$$
$$\times \cdots \hat{\phi}(k_m) \hat{\phi}(-(k_1 + \cdots + k_m)) f(n - \varepsilon_1 k_1 - \cdots - \varepsilon_m k_m), \quad (3.6)$$

where the sum carries over ε_i's belonging to $\{0, 1\}$, and $\tilde{\varepsilon}_i \equiv 1 - \varepsilon_i$.

Expressions of the form (3.6) will naturally come up in our asymptotic analysis of the low-density limit in (2.2), see Sec. 5. Surprisingly enough, such expressions appeared already in refs. 6 and 7, in a quite different context, namely in the study of the behaviour of a *single* electron in a *box* which is either periodic, or complemented with zero boundary conditions (walls). In a low density regime, these works prove the convergence of the underlying Schrödinger equation towards a linear Boltzmann equation with a cross-section given by the Q_m's in (3.6), which is next proved to coincide with the usual Born series expansion. This is the result we recall here. As the reader may easily check, the following link between Q_m and Σ_m is non trivial. It will play a key role in the sequel.

Property (Borrowed from ref. 7). *The following equality holds, whenever f is a smooth function:*

$$Q_m f(n) \equiv 2\pi \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \, \delta(n^2 - k^2) \, \Sigma_m(n, k) \, [f(k) - f(n)], \quad (3.7)$$

In other words, the somehow strange collision term Q_m in (3.6) involves a cross section that coincides *with the m-th order term of the Born series Σ_m. In particular, upon summing over m, we recover*

$$\sum_{m \geq 1} Q_m f(n) \equiv 2\pi \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \, \delta(n^2 - k^2) \, \Sigma^{\text{low}}(n, k) \, [f(k) - f(n)], \quad (3.8)$$

for any smooth function f. Hence the Q_m's allow to build up the Born series expansion after summation.

Remark. *The identity (3.7) implies the following: for the gain term, we have*

$$\frac{2\pi}{(2\pi)^3} \delta(n^2 - k^2) \, \Sigma_m(n, k) = 2 \text{ Re } \frac{(-i)^{m+1}}{(2\pi)^{3m}} \sum_{(\varepsilon_1,\ldots,\varepsilon_m) \neq (0,\ldots,0)} (-1)^{\varepsilon_1+\cdots+\varepsilon_m}$$

$$\times \int dk_1 \ldots dk_m \Delta(n - \varepsilon_1 k_1, n + \tilde{\varepsilon}_1 k_1) \cdots \Delta(n - \varepsilon_1 k_1 - \cdots - \varepsilon_m k_m, n + \tilde{\varepsilon}_1 k_1 + \cdots + \tilde{\varepsilon}_m k_m) \hat{\phi}(k_1) \cdots \hat{\phi}(k_m) \hat{\phi}(-(k_1 + \cdots + k_m)) \times \delta(k - [n - \varepsilon_1 k_1 - \cdots - \varepsilon_m k_m]), \quad (3.9)$$

whereas for the loss term, there holds

$$\frac{2\pi}{(2\pi)^3} \int dk\, \delta(n^2 - k^2)\, \Sigma_m(n, k) = 2\, \mathrm{Re}\, \frac{(-i)^{m+1}}{(2\pi)^{3m}} \int dk_1 \ldots dk_m \Delta(n, n + k_1) \cdots \Delta(n, n + k_1 + \cdots + k_m)\, \hat{\phi}(k_1) \cdots \hat{\phi}(k_m) \hat{\phi}(-(k_1 + \cdots + k_m)). \quad (3.10)$$

Remark. *Without giving the actual proof of the identity (3.8), we here give the basic ingredient at the origin of this formula, in the case $m = 2$. On the one hand, and as the reader may easily check, the term Q_2 involves the quantity (after easy changes of variables)*

$$\mathrm{I} = \mathrm{Re}\, i \big[- \big[\Delta(k, n)\, \Delta(k, k_1) + \Delta(n, k_1)\, \Delta(k, k_1)\big]\, \hat{\phi}(n - k) \hat{\phi}(k - k_1) \times \hat{\phi}(k_1 - n) + \Delta(k_1, n)\, \Delta(k, n)\, \hat{\phi}(n - k_1) \hat{\phi}(k_1 - k) \hat{\phi}(k - n)\big], \quad (3.11)$$

while Σ_2 involves

$$\mathrm{II} = i\, \big[- \Delta(n, k_1)\, \hat{\phi}(n - k) \hat{\phi}(k - k_1) \hat{\phi}(k_1 - n) + \Delta(k_1, n)\, \hat{\phi}(n - k_1) \hat{\phi}(k_1 - k) \hat{\phi}(k - n)\big]. \quad (3.12)$$

Now, the basic identity is obtained upon writing $\Delta(n, p) = i/(n^2 - p^2 + i0)$:

$$\Delta\,(k, k_1)\, [\Delta\,(k, n) + \Delta\,(n, k_1)] = \Delta\,(k, n)\ \ \Delta\,(n, k_1)\,. \quad (3.13)$$

This, together with the fact that $\mathrm{Re}\Delta(n, p) = \pi\delta(n^2 - p^2)$*, establishes the equality between the two quantities* I *and* $\pi\delta(n^2 - k^2) \times \mathrm{II}$.

Formula (3.7) is roughly obtained upon iterating this argument in the appropriate recursion formula.

4. ANALYSIS OF $S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}$: UNIFORM BOUNDS

As explained in detail in Sec. 2.4, the present section and the next one are devoted to the careful study of the reference operator $S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}$, for any given, fixed values of the integers a, b, and m (operator $C_{a,b}^{m,\varepsilon}$ is defined by (2.32)). We refer to this section for the reason why we dwell on that particular term. The present Sec. 4 is actually devoted to deriving uniform bounds on $S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}$. Our final estimate is given in Theorem 2.

We may first write, using (2.32) and recalling the abuse of notation (2.24),

$$S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2} = \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m-1}} d\tau_m \\ \times\, S(t-t_1)\,\tilde{C}_{a,b}^{\varepsilon}\,S(t_1-\tau_1)\,\tilde{T}_{a,b}^{\varepsilon}\,S(\tau_1-\tau_2)\,\tilde{T}_{a,b}^{\varepsilon} \\ \times \cdots S(\tau_{m-1}-\tau_m)\,\tilde{T}_{a,b}^{\varepsilon}\,S(\tau_m)\,(f_0\otimes f_0)\,. \tag{4.1}$$

The operators $\tilde{C}_{a,b}^{\varepsilon}$ and $\tilde{T}_{a,b}^{\varepsilon}$ are defined in (2.14) and (2.11) respectively. They involve both diverging prefactors ε^{-1} and $N-1\approx\varepsilon^{-2}$, together with strong oscillatory terms of the form $\exp(ih\cdot x/\varepsilon)$, see (2.11) and (2.14). The uniform bounds on (4.1) to be derived are obtained in two steps. As a first step (Sec. 4.1), we rewrite $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$ in a form that is convenient for the analysis. This is the crucial step: we carefully analyse the above mentionned oscillatory phases to balance the diverging prefactors, and obtain a bounded, $O(1)$ contribution. Our final result in that direction is formula (4.7). In a second step (Sec. 4.2), we use the so-obtained formula (4.7) to get bounds on $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$, that are uniform in ε. This second step uses the stationary phase formula, together with an estimate on the size of the determinant of some quadratic form (Lemma 1). The final result is given in Sec. 4.3.

4.1. Rewriting $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$ in a Convenient form

Going back to formulae (2.11) and (2.14) that give the value of the operators $\tilde{T}_{a,b}^{\varepsilon}$ and $\tilde{C}_{a,b}^{\varepsilon}$, we first observe that $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$ defined in (4.1) has the explicit value

$$\big(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\big)(t,x_a,v_a) = (-i)^{m+1}\,\frac{N-1}{\varepsilon^m}\sum_{\sigma_1,\ldots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{t_1} d\tau_1 \\ \times \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m-1}} d\tau_m \int_{\mathbb{R}^{3(m+1)+6}} dx_b\,dv_b\,\frac{dk_1}{(2\pi)^3}\cdots\frac{dk_{m+1}}{(2\pi)^3}\;\hat{\phi}(k_1)\cdots\hat{\phi}(k_{m+1}) \\ \times \exp\left(\frac{i}{\varepsilon}k_1[x_a - v_a(t-t_1) - x_b]\right) \\ \times \exp\left(\frac{i}{\varepsilon}k_2[x_a - v_a(t-t_1) - x_b - (t_1-\tau_1)(v_a - v_b - \sigma_1 k_1)]\right)\cdots\cdots \\ \times \exp\Bigg(\frac{i}{\varepsilon}k_{m+1}[x_a - v_a(t-t_1) - x_b - (t_1-\tau_1)(v_a-v_b-\sigma_1 k_1) - (\tau_1-\tau_2) \\ \times (v_a - v_b - \sigma_1 k_1 - \sigma_2 k_2) - \cdots - (\tau_{m-1}-\tau_m)$$

$$\times \, (v_a - v_b - \sigma_1 k_1 - \cdots - \sigma_m k_m)] \Big) f_0(x_a(0), v_a(0)) \; f_0(x_b(0), v_b(0)). \qquad (4.2)$$

Here the σ_i's take the value ± 1. Also, the initial positions $x_a(0)$ and $x_b(0)$ are given by

$$\begin{aligned} x_a(0) &= x_a - v_a(t-t_1) - (t_1-\tau_1)\left(v_a - \sigma_1\frac{k_1}{2}\right) - (\tau_1-\tau_2) \\ &\quad \times \left(v_a - \sigma_1\frac{k_1}{2} - \sigma_2\frac{k_2}{2}\right) - \cdots - (\tau_{m-1}-\tau_m) \\ &\quad \times \left(v_a - \sigma_1\frac{k_1}{2} - \cdots - \sigma_m\frac{k_m}{2}\right) - \tau_m\left(v_a - \sigma_1\frac{k_1}{2} - \cdots - \sigma_{m+1}\frac{k_{m+1}}{2}\right), \\ x_b(0) &= x_b - (t_1-\tau_1)\left(v_b + \sigma_1\frac{k_1}{2}\right) - (\tau_1-\tau_2) \\ &\quad \times \left(v_b + \sigma_1\frac{k_1}{2} + \sigma_2\frac{k_2}{2}\right) - \cdots - (\tau_{m-1}-\tau_m) \\ &\quad \times \left(v_b + \sigma_1\frac{k_1}{2} + \cdots + \sigma_m\frac{k_m}{2}\right) - \tau_m\left(v_b + \sigma_1\frac{k_1}{2} + \cdots + \sigma_{m+1}\frac{k_{m+1}}{2}\right), \end{aligned} \qquad (4.3)$$

and the initial velocities $v_a(0)$ and $v_b(0)$ satisfy

$$\begin{aligned} v_a(0) &= v_a - \sigma_1\frac{k_1}{2} - \cdots - \sigma_{m+1}\frac{k_{m+1}}{2}, \\ v_b(0) &= v_b + \sigma_1\frac{k_1}{2} + \cdots + \sigma_{m+1}\frac{k_{m+1}}{2}\,. \end{aligned} \qquad (4.4)$$

Besides, recall that the low density scaling imposes $N = \varepsilon^{-2}$, so that we may safely replace $(N-1)/\varepsilon^m$ by the simpler ε^{-m-3} in formula (4.2). As a consequence, and up to this abuse of notation, we obtain, putting some terms together:

$$\begin{aligned} &\left(S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}\right)(t, x_a, v_a) \\ &= (-i)^{m+1}\,\varepsilon^{-m-3} \sum_{\sigma_1,\dots,\sigma_{m+1}} \sigma_1 \cdots \sigma_{m+1} \int_0^t dt_1 \int_0^{t_1} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{m-1}} d\tau_m \\ &\times \int_{\mathbb{R}^{3(m+1)+6}} dx_b\, dv_b\, \frac{dk_1}{(2\pi)^3} \cdots \frac{dk_{m+1}}{(2\pi)^3}\;\; \hat{\phi}(k_1)\cdots\hat{\phi}(k_{m+1}) \\ &\times \exp\left(\frac{i}{\varepsilon}(k_1 + \cdots + k_{m+1})(x_a - v_a(t-t_1) - x_b)\right) \end{aligned}$$

$$\times\ \exp\left(-\frac{i}{\varepsilon}(t_1-\tau_1)(k_2+\cdots+k_{m+1})(v_a-v_b-\sigma_1 k_1)\right)$$
$$\times\ \exp\left(-\frac{i}{\varepsilon}(\tau_1-\tau_2)(k_3+\cdots+k_{m+1})(v_a-v_b-\sigma_1 k_1-\sigma_2 k_2)\right)\cdots\cdots$$
$$\times\ \exp\left(-\frac{i}{\varepsilon}(\tau_{m-1}-\tau_m)k_{m+1}(v_a-v_b-\sigma_1 k_1-\cdots-\sigma_m k_m)\right)$$
$$f_0(x_a(0), v_a(0))\ f_0(x_b(0), v_b(0)), \tag{4.5}$$

with $(x_a(0), v_a(0))$ and $(x_b(0), v_b(0))$ still given by (4.3)–(4.4).

In order to get bounds on (4.5) and to identify the limit later, one actually needs to go to more convenient variables. In view of (4.5), it is natural to set

$$t_1-\tau_1=\varepsilon s_1\ ,\ \cdots\ ,\ \tau_{m-1}-\tau_m=\varepsilon s_m,$$
$$togetherwith\ \ K_1=k_1, K_2=k_2,\dots,K_m=k_m, \xi_b=(k_1+\cdots+k_{m+1})/\varepsilon. \tag{4.6}$$

It also turns out to be convenient to take as new variables the *relative* position and velocities

$$X_b=x_b-[x_a-v_a(t-t_1)],\ \ K_0=v_b-v_a,$$

instead of (x_b, v_b) in (4.5), to treat the $dx_b dv_b$ integration.

Grouping terms in the appropriate way, all this gives our final expression of $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$:

$$\begin{aligned}
&\left(S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}\right)(t, x_a, v_a)\\
&\quad=(-i)^{m+1}\sum_{\sigma_1,\dots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{t_1/\varepsilon}ds_1\int_0^{t_1/\varepsilon-s_1}ds_2\cdots\int_0^{t_1/\varepsilon-s_1-\cdots-s_{m-1}}ds_m\\
&\quad\times\int_{\mathbb{R}^{3(m+1)+6}}dX_b\,d\xi_b\,\frac{dK_0}{(2\pi)^3}\cdots\frac{dK_m}{(2\pi)^3}\ \hat\phi(K_1)\cdots\hat\phi(K_m)\,\hat\phi(-(K_1+\cdots+K_m)\\
&\quad+\varepsilon\xi_b)\exp(-iK\cdot QK)\ \exp(-iX_b\cdot\xi_b+i\varepsilon\ \xi_b\cdot MK)\\
&\quad\times\ f_0\,(x_a-v_a t+[RKt_1+\varepsilon NK+\varepsilon\mu\xi_b], v_a-[RK+\varepsilon l\xi_b])\,f_0(X_b+x_a\\
&\quad\times\ -v_a t-K_0 t_1-[RKt_1+\varepsilon NK+\varepsilon\mu\xi_b], K_0+v_a+[RK+\varepsilon\lambda\xi_b]).
\end{aligned}\tag{4.7}$$

This is the form we use from now on to get bounds on $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$, and to compute its limiting value.

Formula (4.7) involves the following short-hand notations. First, we have set the vector $K=(K_0,\dots,K_m)$ for convenience. Second, the symmetric matrix Q, which is the truly important term for all the subsequent analysis, is defined by the quadratic form

$$K \cdot QK := s_1 K_1 (K_0 + \sigma_1 K_1) + s_2 (K_1 + K_2)(K_0 + \sigma_1 K_1 + \sigma_2 K_2) + \cdots \\ + s_m (K_1 + \cdots + K_m)(K_0 + \sigma_1 K_1 + \cdots + \sigma_m K_m). \tag{4.8}$$

Note in passing that Q has coefficients depending on the σ_i's, and the s_i's in (4.7), but we do not emphasize this dependence for convenience. Last, we have set the auxiliary matrices M, N, R, as well as the auxiliary scalars λ and μ, as

$$\begin{aligned} RK &:= \frac{\sigma_1 - \sigma_{m+1}}{2} K_1 + \frac{\sigma_2 - \sigma_{m+1}}{2} K_2 + \cdots + \frac{\sigma_m - \sigma_{m+1}}{2} K_m, \\ \varepsilon MK &:= \varepsilon s_1 (K_0 + \sigma_1 K_1) + \varepsilon s_2 (K_0 + \sigma_1 K_1 + \sigma_2 K_2) \\ &\quad + \cdots + \varepsilon s_m (K_0 + \sigma_1 K_1 + \cdots + \sigma_m K_m), \\ \varepsilon \lambda \xi_b &:= \varepsilon \sigma_{m+1} \frac{\xi_b}{2}, \\ \varepsilon NK &:= \varepsilon s_1 (\sigma_1 K_1) + \varepsilon s_2 (\sigma_1 K_1 + \sigma_2 K_2) + \cdots + \varepsilon s_m (\sigma_1 K_1 \\ &\quad + \cdots + \sigma_m K_m) - \varepsilon RK (s_1 + \cdots + s_m), \\ \varepsilon \mu \xi_b &:= \varepsilon \lambda \xi_b (t_1 - \varepsilon [s_1 + \cdots + s_m]). \end{aligned} \tag{4.9}$$

Note that only R corresponds to an order one contribution in (4.7), while M, N, λ and μ induce a vanishing, $O(\varepsilon)$ effect. Note also that RK is an "impulse variable" (as well as $\lambda\xi_b$ and MK), while NK is a "position variable" (as well as $\mu\xi_b$). Again, we do not emphasize the dependence of the matrices M, N, R, as well as the scalars λ and μ, upon the σ_i's, and the s_i's.

Summarizing, when going from the original expression (4.2) to the final (4.7), the diverging prefactor ε^{-m-3} has been absorbed through the changes of variables (4.6). As a consequence, the difficulty has been transfered into the question of getting *integrability* in the variables $s_1, \ldots, s_m$ close to $+\infty$ in formula (4.7), because the upper bounds $t_1/\varepsilon - s_1$ and so on in (4.7) now go to $+\infty$ as $\varepsilon \to 0$. This integrability property will come from the crucial oscillatory factor $\exp(iK \cdot QK)$ in (4.7).

4.2. Obtaining Uniform Bounds on $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$

We now derive bounds on $S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}$ with the help of formula (4.7). It turns out that bounds are more easily derived on the *Fourier transform* of this term: natural bounds in L^1 and L^∞ are indeed to be obtained in the Fourier space.

We define the Fourier transform

$$\mathcal{F}\left(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\right)(t, \xi_a, \eta_a) := \mathcal{F}_{x_a, v_a}\left(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\right)(t, \xi_a, \eta_a).$$

In view of (4.7), we readily have the explicit value,

$$
\begin{aligned}
\mathcal{F}\left(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a) &= (-i)^{m+1}\sum_{\sigma_1,\ldots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{t_1/\varepsilon} ds_1 \\
&\times\int_0^{t_1/\varepsilon-s_1} ds_2\cdots\int_0^{t_1/\varepsilon-s_1-\cdots-s_{m-1}} ds_m\int_{\mathbb{R}^{3(m+1)+12}} dx_a\,dv_a\,dX_b\,d\xi_b\,\frac{dK_0}{(2\pi)^3}\cdots \\
&\times\frac{dK_m}{(2\pi)^3}\,\hat{\phi}(K_1)\cdots\hat{\phi}(K_m)\hat{\phi}(-(K_1+\cdots+K_m)+\varepsilon\xi_b) \\
&\times\exp(-iK\cdot QK)\,\exp(-iX_b\cdot\xi_b-ix_a\xi_a-iv_a\eta_a+i\varepsilon\,\xi_b\cdot MK) \\
&\times f_0\,(x_a-v_at+[RKt_1+\varepsilon NK+\varepsilon\mu\xi_b],v_a-[RK+\varepsilon\lambda\,\xi_b]) \\
&\times f_0(X_b+x_a-v_at-K_0t_1-[RKt_1+\varepsilon NK+\varepsilon\mu\xi_b],K_0+v_a \\
&+[RK+\varepsilon l\xi_b]).
\end{aligned}
\tag{4.10}
$$

Explicitely performing the $dx_a dv_a dX_b$ integration in (4.10), and putting the important phase factors apart, gives the simpler

$$
\begin{aligned}
\mathcal{F}\left(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a) &= (-i)^{m+1}\sum_{\sigma_1,\ldots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{t_1/\varepsilon} ds_1 \\
&\times\int_0^{t_1/\varepsilon-s_1} ds_2\cdots\int_0^{t_1/\varepsilon-s_1-\cdots-s_{m-1}} ds_m\int_{\mathbb{R}^{3(m+1)+6}} d\xi_b\,d\eta_b\,\frac{dK_0}{(2\pi)^3}\cdots\frac{dK_m}{(2\pi)^3} \\
&\times\hat{\phi}(K_1)\cdots\hat{\phi}(K_m)\,\hat{\phi}(-(K_1+\cdots+K_m)+\varepsilon\xi_b) \\
&\times\,\exp(-iK\cdot QK)\,\exp(iK_0\cdot(\eta_b-\xi_bt_1)+\varepsilon\,\xi_b\cdot MK) \\
&\times\,\exp\big(i(RKt_1+\varepsilon NK+\varepsilon\mu\xi_b)\cdot(\xi_a-2\xi_b) \\
&+\,i(RK+\varepsilon\lambda\,\xi_b)\cdot(2\eta_b-\eta_a-\xi_at)\big)\hat{f}_0(\xi_a-\xi_b,\eta_a+\xi_at-\eta_b)\,\hat{f}_0(\xi_b,\eta_b).
\end{aligned}
\tag{4.11}
$$

In other words,

$$
\begin{aligned}
\mathcal{F}\left(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a) =: \int_0^t dt_1\int_0^{t_1/\varepsilon} ds_1\int_0^{t_1/\varepsilon-s_1} ds_2\,\cdots \\
\times\int_0^{t_1/\varepsilon-s_1-\cdots-s_{m-1}} ds_m B(s_1,\ldots,s_m;\xi_a,\eta_a).
\end{aligned}
\tag{4.12}
$$

Expressions (4.11) together with (4.12) serve as a definition of the quantity $B(s_1,\ldots,s_m;\xi_a,\eta_a)$. We do not emphasize the dependence of B upon the other parameters, like a, b, m, t, t_1, etc.

We are now in position to obtain bounds $\mathcal{F}(S\,C_{a,b}^{m,\varepsilon}\,S\,f_0^{\otimes 2})$ when written in the form (4.11). The strategy we adopt below, splitted in Secs. 4.2.1 to 4.2.3, is to prove that the quantity $B(s_1,\ldots,s_m;\xi_a,\eta_a)$ is integrable over the set $(s_1,\ldots,s_m)\in[0,+\infty[^m$, uniformly in ε (with values in $L^1_{\xi_a,\eta_a}$, essentially):

- for *small* values of the time variables $(s_1,\ldots,s_m)$ in (4.11), we simply perform the dK_0 integration in (4.11). This shows (see (4.14)–(4.15)) that B is bounded in that region.
- for *large* values of the times $(s_1,\ldots,s_m)$ in (4.11), we need to some get decay of B. To do so, we make use of the quadratic form Q, and prove that the complex gaussian $\exp(iK\cdot QK)$ is (weakly) of size $(s_1s_2\ldots s_m)^{-3/2}$ at most (Lemma 1): this requires the use of the Parseval formula in the variables $(K_0,\ldots,K_m)$ in (4.11). As a consequence, we recover that B is an integrable function of $(s_1,\ldots,s_m)$ at infinity, see (4.18)–(4.19).

4.2.1. Small Time Estimates

Explicitely performing the dK_0 integration in (4.11), and using the dependence of both the quadratic form Q and the matrix M in K_0, gives a Dirac mass at

$$\begin{aligned}\eta_b &= s_1K_1+\cdots+s_m(K_1+\cdots+K_m)+\xi_b(t_1-\varepsilon[s_1+\cdots+s_m])\\ &=:\underline{\eta_b}(t_1;s_1,\ldots,s_m;K_1,\ldots,K_m;\xi_b),\end{aligned}$$

in the integration that defines B. This gives the following upper bound,

$$\begin{aligned}&|B(s_1,\ldots,s_m;\xi_a,\eta_a)|\\ &\le\frac{1}{(2\pi)^{m+1}}\sum_{\sigma_1,\cdots,\sigma_{n+1}}\int d\xi_b\,dK_1\cdots dK_m\,\left|\hat\phi(K_1)\cdots\hat\phi(K_m)\hat\phi(-(K_1+\cdots\right.\\ &\left.+K_m)+\varepsilon\xi_b)\right|\,\left|\hat f_0(\xi_a-\xi_b,\eta_a+\xi_at-\underline{\eta_b}(t_1;s_1,\ldots,s_m;\right.\\ &\left.\times\,K_1,\ldots,K_m;\xi_b))\right|\,\left|\hat f_0(\xi_b,\underline{\eta_b}(t_1;s_1,\ldots,s_m;K_1,\ldots,K_m;\xi_b))\right|.\end{aligned}\tag{4.13}$$

From (4.13) we deduce the easy L^1 estimate

$$\left\|B(s_1,\ldots,s_m;\xi_a,\eta_a)\right\|_{L^1_{\xi_a}L^1_{\eta_a}}\le\pi^{-(m+1)}\|\hat\phi\|^m_{L^1}\,\|\hat\phi\|_{L^\infty}\,\|\hat f_0\|_{L^1_{\xi_a}L^1_{\eta_a}}\,\|\hat f_0\|_{L^1_{\xi_a}L^\infty_{\eta_a}}.\tag{4.14}$$

Here we used the fact that the cardinality of the σ_i's in the sum in (4.11) is 2^{m+1}.

Also, due to the appearance of the $L^1_{\xi_a}L^\infty_{\eta_a}$ norm of $\hat f_0$ on the right-hand-side of (4.14), and in order to perform a fixed point procedure later, one needs to write an $L^1_{\xi_a} L^\infty_{\eta_a}$ bound on B as well. Eq. (4.13) allows to write the easy

$$\left\| B(s_1, \ldots, s_m; \xi_a, \eta_a) \right\|_{L^1_{\xi_a} L^\infty_{\eta_a}} \leq \pi^{-(m+1)} \|\hat{\phi}\|^m_{L^1} \|\hat{\phi}\|_{L^\infty} \|\hat{f}_0\|^2_{L^1_{\xi_a} L^\infty_{\eta_a}}. \tag{4.15}$$

4.2.2. Large Time Estimates

For large values of the s_i's, we may first use Parseval formula in the variables $(K_0, \ldots, K_m)$ in the definition of B (see (4.11)). Let $\alpha = (\alpha_0, \ldots, \alpha_m)$ be the Fourier variable. This procedure shows that the integral defining B has the value

$$B(s_1, \ldots, s_m; \xi_a, \eta_a) = \sum_{\sigma_1, \ldots, \sigma_{m+1}} \frac{(2\pi)^{-(m+1)/2}}{\det(Q)^{1/2}}$$
$$\times \int d\eta_b \, d\xi_b \, d\alpha_0 \ldots d\alpha_m \, \hat{f}_0(\xi_a - \xi_b, \eta_a + \xi_a t - \eta_b) \, \hat{f}_0(\xi_b, \eta_b)$$
$$\times \exp(i\alpha \cdot Q^{-1}\alpha) \mathcal{F}_{K_0, \ldots, K_m} (\hat{\phi}(K_1) \cdots \hat{\phi}(K_m) \, \hat{\phi}(-(K_1 + \cdots + K_m)$$
$$+ \varepsilon\xi_b) \exp(iK_0 \cdot (\eta_b - \xi_b t_1) + \varepsilon \, \xi_b \cdot MK) \exp(i(RKt_1 + \varepsilon NK$$
$$+ \varepsilon\mu\xi_b) \cdot (\xi_a - 2\xi_b) + i(RK + \varepsilon\lambda\xi_b) \cdot (2\eta_b - \eta_a - \xi_a t)))(\alpha_0, \ldots, \alpha_n). \tag{4.16}$$

Thus, using that the phase factor inside the Fourier transform in (4.16) depends *linearly* in $(K_0, \ldots, K_m)$, we obtain the bound

$$\left| B(s_1, \ldots, s_m; \xi_a, \eta_a) \right| \leq \frac{c_0^{m+1} \, \|\phi\|^{m+1}_{L^1}}{|\det(Q)|^{1/2}} \int d\eta_b \, d\xi_b \left| \hat{f}_0(\xi_a - \xi_b, \eta_a + \xi_a t - \eta_b) \right| \left| \hat{f}_0(\xi_b, \eta_b) \right|, \tag{4.17}$$

for some universal constant c_0. Now using Lemma 1 below, we deduce the easy two bounds

$$\left\| B(s_1, \ldots, s_m; \xi_a, \eta_a) \right\|_{L^1_{\xi_a} L^1_{\eta_a}} \leq c_0^{m+1} \, (s_1 \cdots s_m)^{-3/2} \, \|\phi\|^{m+1}_{L^1} \, \|\hat{f}_0\|^2_{L^1_{\xi_a} L^1_{\eta_a}}, \tag{4.18}$$

and,

$$\left\| B(s_1, \ldots, s_m; \xi_a, \eta_a) \right\|_{L^1_{\xi_a} L^\infty_{\eta_a}} \leq c_0^{m+1} \, (s_1 \cdots s_m)^{-3/2} \, \|\phi\|^{m+1}_{L^1} \times \|\hat{f}_0\|_{L^1_{\xi_a} L^1_{\eta_a}} \|\hat{f}_0\|_{L^1_{\xi_a} L^\infty_{\eta_a}}, \tag{4.19}$$

for some universal constant c_0.

4.2.3. Estimating the Determinant det Q

To conclude this paragraph (see (4.17) and (4.18)–(4.19)), there only remains to state and prove the following

Lemma 1. *Let Q be the quadratic form over $\mathbb{R}^{3(m+1)}$:*

$$K \cdot QK := s_1K_1(K_0 + \sigma_1K_1) + s_2(K_1 + K_2)(K_0 + \sigma_1K_1 + \sigma_2K_2) + \cdots \\ + s_m(K_1 + \cdots + K_m)(K_0 + \sigma_1K_1 + \cdots + \sigma_mK_m).$$

Then, we have,

$$|\det Q| \geq 4^{-3(m+1)}\,(s_1 \cdots s_m)^3. \tag{4.20}$$

Proof of Lemma 1. Using the identity,

$$ab = \frac{1}{4}\left([a+b]^2 - [a-b]^2\right),$$

one writes the quadratic form Q in the following way:

$$4K \cdot QK = s_1([K_0 + (\sigma_1+1)K_1]^2 - [K_0 + (\sigma_1-1)K_1]^2) + s_2([K_0 \\ + (\sigma_1+1)K_1 + (\sigma_2+1)K_2]^2 - [K_0 + (\sigma_1-1)K_1 - (\sigma_2+1)K_2]^2) \\ + \cdots + s_m\big([K_0 + (\sigma_1+1)K_1 + \cdots + (\sigma_m+1)K_m]^2 \\ - [K_0 + (\sigma_1-1)K_1 - \cdots - (\sigma_m+1)K_m]^2\big).$$

Now changing variables, for $i \geq 1$,

$$\tilde{K}_i := \begin{cases} K_0 + (\sigma_1+1)K_1 + \cdots + (\sigma_i+1)K_i & \text{if } \sigma_i = 1, \\ K_0 + (\sigma_1-1)K_1 + \cdots + (\sigma_i-1)K_i & \text{if } \sigma_i = -1, \end{cases}$$

readily transforms Q into

$$4K \cdot QK = \varepsilon_0\tilde{K}_0^2\left(s_1 + \sum_{i\geq 2}\lambda_{0,i}\,s_i\right) + \varepsilon_1\tilde{K}_1^2\left(s_1 + \sum_{i\geq 2}\lambda_{1,i}\,s_i\right) \\ + \varepsilon_2\tilde{K}_2^2\left(s_2 + \sum_{i\geq 3}\lambda_{2,i}s_i\right) + \cdots + \varepsilon_m\tilde{K}_m^2 s_m,$$

where for each $i \geq 0$, the quantity $\varepsilon_i = \pm 1$ depends upon the value of the σ's, and for each i, j, the quantity $\lambda_{i,j} \in \{0, 1\}$ depends and the σ's as well. The Lemma follows. □

4.3. Final Estimate

From formula (4.11), together with the four bounds (4.14), (4.15), (4.18), (4.19), one infers the

Theorem 2. *Let a, b, and m be given integers.*
(i) the following bound holds true,

$$N_1\left[S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}\right] \leq c_0^{m+1}\, N_0(\phi)^{m+1}\, N_1\,(f_0)^2 \int_0^t dt_1\,, \tag{4.21}$$

for some universal constant c_0. Here, $N_0(\phi)$ denotes the norm

$$N_0(\phi) := \|\phi\|_{L^1} + \|\hat{\phi}\|_{L^1} + \|\hat{\phi}\|_{L^\infty}, \tag{4.22}$$

and $N_1\,(f_0)$ denotes the norm

$$N_1\,(f_0) := \left\|\hat{f}_0\right\|_{L^1_{\xi_a} L^1_{\eta_a}} + \left\|\hat{f}_0\right\|_{L^1_{\xi_a} L^\infty_{\eta_a}}. \tag{4.23}$$

(ii) As an immediate consequence, let $j \geq 1$ be any integer. The following bound holds true as well,

$$N_1\left[S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}\right] \leq c_0^{m+1}\, N_0(\phi)^{m+1}\, N_1\,(f_0)^{j+1} \int_0^t dt_1\,, \tag{4.24}$$

for the same universal constant c_0 as in (i). Here, in analogy with (4.23) above, the norm $N_1(g)$ is defined for any function $g(x_1, v_1, \ldots, x_j, v_j)$, as

$$N_1(g) := \|\hat{g}\|_{L^1_{\xi_1,\ldots,\xi_j} L^1_{\eta_1,\ldots,\eta_j}} + \|\hat{g}\|_{L^1_{\xi_1,\ldots,\xi_j} L^\infty_{\eta_1,\ldots,\eta_j}}. \tag{4.25}$$

Theorem 2 provides the desired uniform estimate on $S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}$. It is the final result of the present Sec. 4.

We wish to stress two points in the above bound (4.21) (or equivalently (4.24)). First, it has geometric growth with m, growing like $c_0^m N_0(\phi)^m$. Since we will eventually need to sum up terms of the form $\sum_{m\geq 1} S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}$ in the sequel (see Sec. 5.2, see also Sec. 2.4)), such a feature is crucial in making the corresponding series converge, provided $N_0(\phi)$ is small (see Theorem 4 below). Second, we keep the notation $\int_0^t dt_1$ in (4.21). The reason is that we will eventually need to *iterat*e bounds of the form (4.21). Keeping the term $\int_0^t dt_1$ gives, after the iteration step, iterated integrals of the form $\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n = t^n/n!$. Needless to say, the $1/n!$ will play an important role in getting enough summability in n.

5. ANALYSIS OF $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$: IDENTIFICATION OF THE LIMIT

In this section we prove the identity

$$S\,C^{m}_{a,b}\,S\,f_0^{\otimes 2} = \lim_{\varepsilon\to 0} S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}, \tag{5.1}$$

where the operator $C^{m}_{a,b}$ has been defined in (2.19). In other words, we identify the role of the $(m+1)$-th order cross-section Σ_m that enters the Born series expansion (2.6), in the limit $\lim_{\varepsilon\to 0} S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$. This section is thus complementary to the previous one. The task we now perform combines formula (4.7) obtained in the previous section, together with formula (3.8) recalled before.

5.1. Rewriting the Formulae

Formula (4.7) gives a convenient expression for the term $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$ in (4.1). On the more, estimate (4.21) established in the previous section shows that $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$ is uniformly bounded. The four bounds (4.14), (4.15), (4.18) and (4.19) even allow to pass to the limit in (4.7) using the Lebesgue theorem (in the variables $s_1, \ldots, s_m$, see (4.11)). As a consequence, we readily have the explicit limiting value

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}\right)(t, x_a, v_a) : \\
=\quad & (-i)^{m+1}\sum_{\sigma_1,\ldots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{+\infty} ds_1\int_0^{+\infty} ds_2\cdots\int_0^{+\infty} ds_m \\
&\times\int_{\mathbb{R}^{3(m+1)+6}} dX_b\,d\xi_b\,\frac{dK_0}{(2\pi)^3}\cdots\frac{dK_m}{(2\pi)^3}\;\hat{\phi}(K_1)\cdots\hat{\phi}(K_m) \\
&\times\hat{\phi}(-(K_1+\cdots+K_m))\exp\left(-iK\cdot QK\right)\;\exp\left(-iX_b\cdot\xi_b\right) \\
&\times f_0(x_a - v_a t + RKt_1, v_a - RK)f_0(X_b + x_a - v_a t \\
&- K_0 t_1 - RKt_1, K_0 + v_a + RK).
\end{aligned}\tag{5.2}$$

The matrices Q and R have been defined earlier in (4.8) and (4.9). Note that the limit in (5.2) holds pointwise in t, in the N_1-norm of both sides (the norm N_1 is defined in (4.23) and (4.25)).

To prove the identity (5.1), we first need to transform (5.2) a little bit. Explicitely performing the $d\xi_b$ integration in (5.2) to get a Dirac mass $\delta(X_b)$, we readily obtain,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}\right)(t, x_a, v_a) \\
&= (-i)^{m+1}\sum_{\sigma_1,\ldots,\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}\int_0^t dt_1\int_0^{+\infty} ds_1\int_0^{+\infty} ds_2\cdots\int_0^{+\infty} ds_m
\end{aligned}$$

$$\times \int_{\mathbb{R}^{3(m+1)}} dK_0 \, \frac{dK_1}{(2\pi)^3} \cdots \frac{dK_m}{(2\pi)^3} \quad \exp(-iK \cdot QK) \ \hat{\phi}(K_1) \cdots \hat{\phi}(K_m)$$

$$\times \ \hat{\phi}(-(K_1 + \cdots + K_m)) \ f_0(x_a - v_a t + RKt_1, v_a - RK)$$

$$\times \ f_0(x_a - v_a t - K_0 t_1 - RKt_1, K_0 + v_a + RK). \tag{5.3}$$

In this expression, one reads off the collision of particles a and b at time t_1: they meet at the *same* position $x_a(t_1) = x_a - v_a(t - t_1)$ (this is the meaning of the $\delta(X_b)$), and the collision induces a momentum transfer RK. The cross section associated with this event is given by the factor $\exp(-iK \cdot QK)\,\hat{\phi}(K_1) \cdots \hat{\phi}(-(K_1 + \cdots + K_m))$. Using the value of the matrices Q and R, (5.3) immediately becomes

$$\lim_{\varepsilon \to 0} \left(S \, C_{a,b}^{m,\varepsilon} \, S \, f_0^{\otimes 2} \right) (t, x_a, v_a)$$

$$= (-i)^{m+1} \sum_{\sigma_1, \ldots, \sigma_{m+1}} \sigma_1 \cdots \sigma_{m+1} \int_0^t dt_1 \int_0^{+\infty} ds_1 \int_0^{+\infty} ds_2 \cdots$$

$$\times \int_0^{+\infty} ds_m \int_{\mathbb{R}^{3(m+1)}} dK_0 \, \frac{dK_1}{(2\pi)^3} \cdots \frac{dK_m}{(2\pi)^3}$$

$$\times \ \exp\Big(- i \big[s_1 K_1 (K_0 + \sigma_1 K_1) + \cdots + s_m (K_1 + \cdots + K_m)$$

$$\times \ (K_0 + \sigma_1 K_1 + \cdots + \sigma_m K_m) \big] \Big) \hat{\phi}(K_1) \cdots \hat{\phi}(K_m) \, \hat{\phi}(-(K_1 + \cdots + K_m))$$

$$\times \ f_0 \Bigg(x_a - v_a t + \left[\frac{\sigma_1 - \sigma_{m+1}}{2} K_1 + \cdots + \frac{\sigma_m - \sigma_{m+1}}{2} K_m \right] t_1,$$

$$\times \ v_a - \left[\frac{\sigma_1 - \sigma_{m+1}}{2} K_1 + \cdots + \frac{\sigma_m - \sigma_{m+1}}{2} K_m \right] \Bigg)$$

$$\times \ f_0 \Bigg(x_a - v_a t - K_0 t_1 - \left[\frac{\sigma_1 - \sigma_{m+1}}{2} K_1 + \cdots + \frac{\sigma_m - \sigma_{m+1}}{2} K_m \right] t_1,$$

$$\times \ K_0 + v_a + \left[\frac{\sigma_1 - \sigma_{m+1}}{2} K_1 + \cdots + \frac{\sigma_m - \sigma_{m+1}}{2} K_m \right] \Bigg). \tag{5.4}$$

The first observation is the following: the above formula apparently is a sum of terms depending on the $(m + 1)$-tuple $(\sigma_1, \ldots, \sigma_{m+1})$. However, due to the symmetry $\hat{\phi}(h)^* = \hat{\phi}(-h)$, it is readily checked that for any given value of $(\sigma_1, \ldots, \sigma_{m+1})$, the term corresponding to $(\sigma_1, \ldots, \sigma_{m+1})$ in (5.4), and the one corresponding to $(-\sigma_1, \ldots, -\sigma_{m+1})$, are exactly complex conjugated. This allows to simplify a bit the above expression and only retain the terms stemming from $\sigma_{m+1} = +1$. The second observation is as in the proof of Lemma 1: one writes $ab = ((a+b)/2)^2 - ((a-b)/2)^2$, to transform all the factors $\exp(isab)$ in (5.4), into true complex gaussians. Eventually, exploiting both facts, one obtains the equality:

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \left(S\, C^{m,\varepsilon}_{a,b}\, S\, f_0^{\otimes 2}\right)(t, x_a, v_a) \\
&= 2\mathrm{Re}\,(-i)^{m+1} \sum_{\sigma_1,\dots,\sigma_m} \sigma_1 \cdots \sigma_m \int_0^t dt_1 \int_0^{+\infty} ds_1 \int_0^{+\infty} ds_2 \cdots \\
&\quad \times \int_0^{+\infty} ds_m \int_{\mathbb{R}^{3(m+1)}} dK_0 \frac{dK_1}{(2\pi)^3} \cdots \frac{dK_m}{(2\pi)^3} \\
&\quad \times \exp\left(is_1 \left(\left[\frac{K_0}{2} + \frac{\sigma_1 - 1}{2} K_1 \right]^2 - \left[\frac{K_0}{2} + \frac{\sigma_1 + 1}{2} K_1 \right]^2 \right)\right) \\
&\quad \times \exp\left(is_2 \left(\left[\frac{K_0}{2} + \frac{\sigma_1 - 1}{2} K_1 + \frac{\sigma_2 - 1}{2} K_2 \right]^2 \right.\right. \\
&\quad \times \left.\left. - \left[\frac{K_0}{2} + \frac{\sigma_1 + 1}{2} K_1 + \frac{\sigma_2 + 1}{2} K_2 \right]^2 \right)\right) \cdots \\
&\quad \times \exp\left(is_m \left(\left[\frac{K_0}{2} + \frac{\sigma_1 - 1}{2} K_1 + \cdots + \frac{\sigma_m - 1}{2} K_m \right]^2 \right.\right. \\
&\quad \times \left.\left. - \left[\frac{K_0}{2} + \frac{\sigma_1 + 1}{2} K_1 + \cdots + \frac{\sigma_m + 1}{2} K_m \right]^2 \right)\right) \\
&\quad \times \hat{\phi}(K_1) \cdots \hat{\phi}(K_m)\, \hat{\phi}(-(K_1 + \cdots + K_m)) \\
&\quad \times f_0 \left(x_a - v_a t + \left[\frac{\sigma_1 - 1}{2} K_1 + \cdots + \frac{\sigma_m - 1}{2} K_m \right] t_1, v_a \right. \\
&\quad \times \left. - \left[\frac{\sigma_1 - 1}{2} K_1 + \cdots + \frac{\sigma_m - 1}{2} K_m \right] \right) \\
&\quad \times f_0 \left(x_a - v_a t - K_0 t_1 - \left[\frac{\sigma_1 - 1}{2} K_1 + \cdots + \frac{\sigma_m - 1}{2} K_m \right] t_1, \right. \\
&\quad \times \left. K_0 + v_a + \left[\frac{\sigma_1 - 1}{2} K_1 + \cdots + \frac{\sigma_m - 1}{2} K_m \right] \right).
\end{aligned}
$$

Now, setting $\varepsilon_i = (1-\sigma_i)/2$, $\tilde{\varepsilon}_i = 1-\varepsilon_i = (1+\sigma_i)/2$, and $\sigma_i = (-1)^{\varepsilon_i}$, so that the sum over the σ_i's becomes a sum over ε_i's belonging to $\{0, 1\}$, we recover:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left(S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}\right)(t, x_a, v_a)\\
&= 2\mathrm{Re}\, \frac{(-i)^{m+1}}{(2\pi)^{3m}} \sum_{\varepsilon_1,\dots,\varepsilon_m} (-1)^{\varepsilon_1+\dots+\varepsilon_m} \int_0^t dt_1 \int_{\mathbb{R}^{3(m+1)}} dK_0\, dK_1 \cdots dK_m\\
&\quad \times \Delta\left(\frac{K_0}{2} - \varepsilon_1 K_1, \frac{K_0}{2} + \tilde{\varepsilon}_1 K_1\right)\\
&\quad \times \Delta\left(\frac{K_0}{2} - \varepsilon_1 K_1 - \varepsilon_2 K_2, \frac{K_0}{2} + \tilde{\varepsilon}_1 K_1 + \tilde{\varepsilon}_2 K_2\right)\cdots\\
&\quad \times \Delta\left(\frac{K_0}{2} - \varepsilon_1 K_1 + \cdots - \varepsilon_m K_m, \frac{K_0}{2} + \tilde{\varepsilon}_1 K_1 + \cdots + \tilde{\varepsilon}_m K_m\right)\\
&\quad \times\, \hat{\phi}(K_1)\cdots\hat{\phi}(K_m)\,\hat{\phi}(-(K_1+\cdots+K_m))\\
&\quad \times\, f_0\big(x_a - v_a t - [\varepsilon_1 K_1 + \cdots + \varepsilon_m K_m]\, t_1, v_a + [\varepsilon_1 K_1 + \cdots + \varepsilon_m K_m]\big)\\
&\quad \times\, f_0\big(x_a - v_a t - K_0 t_1 + \big[\varepsilon_1 K_1 + \cdots + \varepsilon_m K_m\big] t_1, K_0 + v_a\\
&\quad \times\, [\varepsilon_1 K_1 + \cdots + \varepsilon_m K_m]\big).
\end{aligned} \tag{5.5}$$

This is the final formula of the present Sec. 5.1.

5.2. Identification of the Born Series

Armed with (5.5), and using the identification (3.7) recalled in Sec. 3, we readily obtain the identity:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \left(S\, C_{a,b}^{m,\varepsilon}\, S\, f_0^{\otimes 2}\right)(t, x_a, v_a)\\
&= 2\pi \int_0^t dt_1 \int_{\mathbb{R}^6} dK_0\, dK'\, \delta\left(\left[\frac{K_0}{2}\right]^2 - K'^2\right) \Sigma_m\left(\frac{K_0}{2}, K'\right)\\
&\quad \times\left[f_0\left(x_a - v_a t - \frac{K_0}{2} t_1 + K' t_1, v_a + \frac{K_0}{2} - K'\right)\right.\\
&\quad \times f_0\left(x_a - v_a t - \frac{K_0}{2} t_1 - K' t_1, \frac{K_0}{2} + v_a + K'\right)\\
&\quad \left. - f_0\,(x_a - v_a t, v_a)\, f_0\,(x_a - v_a t - K_0 t_1, K_0 + v_a)\right],
\end{aligned}$$

where the m-th order term of the Born series, Σ_m, has been defined in (3.3). In other words, introducing the ingoing and outgoing velocities

$$v_b = v_a + K_0, \quad v'_a = v_a + \frac{K_0}{2} - K', \quad v'_b = v_a + \frac{K_0}{2} + K',$$

we recover the

Theorem 3. *Let a, b, m be given integers.*

(i) The following convergence holds:

$$\mathcal{F}_{x_a,v_a}\left(S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a) \longrightarrow \mathcal{F}_{x_a,v_a}\left(S\,C^{m}_{a,b}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a)$$
$$in\ C^0\left(\mathbb{R}^+_t; L^1\left(\mathbb{R}^6_{\xi_a,\eta_a}\right)\right)\ and\ in\ C^0\left(\mathbb{R}^+_t;^1\left(\mathbb{R}^3_{\xi_a}; L^\infty\left(\mathbb{R}^3_{\eta_a}\right)\right)\right). \tag{5.6}$$

Here the operator $C^m_{a,b}$ has been defined in (2.19), namely,

$$\begin{aligned}
&\left(C^m_{a,b} f_0\right)(x_a,v_a)\\
&= 2\pi \int_{\mathbb{R}^9} dv_b\,dv'_a\,dv'_b\ \ \delta(v_a+v_b-v'_a-v'_b)\ \delta\\
&\quad\times\left(\left[\frac{v_a-v_b}{2}\right]^2-\left[\frac{v'_a-v'_b}{2}\right]^2\right)\Sigma_m\left(\frac{v_b-v_a}{2};\frac{v'_b-v'_a}{2}\right)\\
&\quad\times\left[f_0\left(x_a,v'_a\right)f_0\left(x_a,v'_b\right)-f_0\left(x_a,v_a\right)f_0\left(x_a,v_b\right)\right].
\end{aligned}\tag{5.7}$$

(ii) Let j be any integer. The similar convergence result holds for $S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes j}$.

6. SUMMING OVER m TO RECOVER THE FULL BORN SERIES: ANALYSIS OF THE TERM $S\,C^{\varepsilon}_{a,b}\,S\,f_0^{\otimes 2} = \sum_m S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}$.

As an immediate corrollary of both Theorem 2 and Theorem 3, together with the identity

$$S\,C^{\varepsilon}_{a,b}\,S\,f_0^{\otimes 2} = \sum_m S\,C^{m,\varepsilon}_{a,b}\,S\,f_0^{\otimes 2},$$

and the Born series expansion $\Sigma^{\text{low}} = \sum_{m\geq 1}\Sigma_m$, we recover the

Theorem 4. *Let a and b be any integers. Assume that the potential ϕ is small enough, in that*

$$N_0(\phi) \leq c_1, \tag{6.1}$$

for some small, universal constant $c_1 > 0$. Let c_0 be as in Theorem 2.

(i) The following convergence holds:

$$\mathcal{F}_{x_a,v_a}\left(S\,C^{\varepsilon}_{a,b}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a) \longrightarrow \mathcal{F}_{x_a,v_a}\left(S\,C_{a,b}\,S\,f_0^{\otimes 2}\right)(t,\xi_a,\eta_a)$$
$$in\ C^0\left(\mathbb{R}^+_t;L^1\left(\mathbb{R}^6_{\xi_a,\eta_a}\right)\right)\ and\ in\ C^0\left(\mathbb{R}^+_t;L^1\left(\mathbb{R}^3_{\xi_a};L^\infty\left(\mathbb{R}^3_{\eta_a}\right)\right)\right). \tag{6.2}$$

On the more, the following uniform bound holds true

$$N_1\left[S\,C_{a,b}\,S\,f_0^{\otimes 2}\right] \le \left(\sum_{m\ge 1} c_0^{m+1}\,N_0(\phi)^{m+1}\right) N_1(f_0)^2 \int_0^t dt_1.$$

Here, the operator $C^{\varepsilon}_{a,b}$ has been defined in (2.32). Also, $C_{a,b}$ has been defined in (2.17), namely

$$\begin{aligned}
&(C_{a,b}f_0)(x_a,v_a)\\
&= 2\pi \int_{\mathbb{R}^9} dv_b\,dv'_a\,dv'_b\ \ \delta(v_a+v_b-v'_a-v'_b)\ \delta\\
&\quad\times\left(\left[\frac{v_a-v_b}{2}\right]^2-\left[\frac{v'_a-v'_b}{2}\right]^2\right)\Sigma^{\mathrm{low}}\left(\frac{v_b-v_a}{2};\frac{v'_b-v'_a}{2}\right)\\
&\quad\times\left[f_0\left(x_a,v'_a\right)f_0\left(x_a,v'_b\right)-f_0\left(x_a,v_a\right)f_0\left(x_a,v_b\right)\right].
\end{aligned}\tag{6.3}$$

(ii) Let j be any integer. The similar convergence result holds for $S\,C^{\varepsilon}_{a,b}\,S\,f_0^{\otimes j}$.

Naturally, in view of the bound (4.21), the constant c_1 in Theorem 4 is such that $c_0c_1 < 1$.

7. LINKING TWO COLLISIONS: ANALYSIS OF $S\,C^{\varepsilon}_{R_1,J+1}\,S\,C^{\varepsilon}_{R_2,J+2}\,S\,F_0^{\otimes J+2}$

In the spirit of what we did in Sec. 4 to 6, we now would like to analyse the term

$$S\,C^{\varepsilon}_{r_1,j+1}\,S\,C^{\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}.$$

We wish to prove it converges towards

$$S\,C_{r_1,j+1}\,S\,C_{r_2,j+2}\,S\,f_0^{\otimes j+2},$$

where $C_{r_1,j+1}$ is the classical collision operator defined in (2.17) (it is intended that $r_1 \le j$ and $r_2 \le j+1$). To do so, the analysis is now splitted into two steps. As a first step, for given exponents m and p, we study the term

$$S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}. \tag{7.1}$$

We give uniform bounds in the spirit of Theorem 2, together with a convergence result in the spirit of Theorem 3. Next, we sum over m and p to conclude the paragraph.

7.1. Analysis of $S\, C^{m,\varepsilon}_{r_1,j+1}\, S\, C^{p,\varepsilon}_{r_2,j+2}\, S\, f_0^{\otimes j+2}$

To simplify the writing, we here assume $r_1 \neq r_2$. The analysis in the case $r_1 = r_2$ follows exactly the same lines. In essence, the present paragraph is merely technical, and consists in rephrasing what has already been done in paragraph 4. The only important task here is to check that the independence of variables that lied at the core of the proof in paragraph 4, in the case of two "colliding" particles, still holds in the case when particles r_1 and $j+1$, together with particles r_2 and $j+2$, "collide" in a row.
For later convenience, let us define the indices:

$$a = r_1, b = j+1, c = r_2, d = j+2.$$

Also, as we already took note, it is enough here to investigate the case

$$j = 2.$$

We observe, as we did in (4.2), the equality (here we made the harmless abuse of notation $N-1 = \varepsilon^{-2}$ and so on),

$$\begin{aligned}
&\left(S\, C^{m,\varepsilon}_{a,b}\, S\, C^{p,\varepsilon}_{c,d}\, S\, f_0^{\otimes 4}\right)(t, x_a, v_a, x_c, v_c) \\
&= (-i)^{m+p+2}\, \varepsilon^{-m-p-6} \sum_{\substack{\sigma_1,\ldots,\sigma_{m+1} \\ \sigma'_1,\ldots,\sigma'_{p+1}}} \sigma_1 \cdots \sigma_{m+1} \sigma'_1 \cdots \sigma'_{p+1} \\
&\quad \times \int_0^t dt_1 \int_0^{t_1} d\tau_1 \cdots \int_0^{\tau_{m-1}} d\tau_m \\
&\quad \times \int_0^{\tau_m} dt_2 \int_0^{t_2} d\tau'_1 \cdots \int_0^{\tau'_{p-1}} d\tau'_p \\
&\quad \times \int dx_b\, dv_b\, \frac{dk_1}{(2\pi)^3} \cdots \frac{dk_{m+1}}{(2\pi)^3}\, dx_d\, dv_d\, \frac{dk'_1}{(2\pi)^3} \cdots \\
&\quad \times \frac{dk'_{p+1}}{(2\pi)^3} \quad \hat{\phi}(k_1) \cdots \hat{\phi}(k_{m+1}) \;\; \hat{\phi}(k'_1) \cdots \hat{\phi}(k'_{p+1}) \\
&\quad \times \exp\Bigg(\frac{i}{\varepsilon} k_1 \left[x_a - v_a(t-t_1) - x_b\right] + \cdots + \frac{i}{\varepsilon} k_{m+1} \\
&\quad \times \big[x_a - v_a(t-t_1) - x_b - (t_1 - \tau_1)(v_a - v_b - \sigma_1 k_1)
\end{aligned}$$

$$\times - (\tau_1 - \tau_2)(v_a - v_b - \sigma_1 k_1 - \sigma_2 k_2) - \cdots - (\tau_{m-1} - \tau_m)$$

$$\times (v_a - v_b - \sigma_1 k_1 - \cdots - \sigma_m k_m)\big]\Big)$$

$$\times \exp\left(\frac{i}{\varepsilon} k_1' \left[x_c - v_c(t - t_2) - x_d\right] + \cdots + \frac{i}{\varepsilon} k_{p+1}' \big[x_c - v_c(t - t_2) - x_d\right.$$

$$\times - (t_2 - \tau_1')(v_c - v_d - \sigma_1' k_1') - (\tau_1' - \tau_2')\big(v_c - v_d - \sigma_1' k_1' - \sigma_2' k_2'\big) - \cdots$$

$$\left.\times - (\tau_{p-1}' - \tau_p')\big(v_c - v_d - \sigma_1' k_1' - \cdots - \sigma_p' k_p'\big)\big]\right)$$

$$\times f_0(x_a(0), v_a(0))\; f_0(x_b(0), v_b(0))\; f_0(x_c(0), v_c(0))\; f_0(x_d(0), v_d(0)).$$

Here the σ_i's and σ_i''s take the value ± 1. Also, the initial positions $x_a(0)$, $x_b(0)$, $x_c(0)$ and $x_d(0)$ are given by

$$x_a(0) = x_a - v_a(t - t_1) - (t_1 - \tau_1)\left(v_a - \sigma_1 \frac{k_1}{2}\right) - (\tau_1 - \tau_2)$$

$$\times \left(v_a - \sigma_1 \frac{k_1}{2} - \sigma_2 \frac{k_2}{2}\right) - \cdots - (\tau_{m-1} - \tau_m)$$

$$\times \left(v_a - \sigma_1 \frac{k_1}{2} - \cdots - \sigma_m \frac{k_m}{2}\right) - \tau_m \left(v_a - \sigma_1 \frac{k_1}{2} - \cdots - \sigma_{m+1} \frac{k_{m+1}}{2}\right),$$

$$x_b(0) = x_b - (t_1 - \tau_1)\left(v_b + \sigma_1 \frac{k_1}{2}\right) - (\tau_1 - \tau_2)\left(v_b + \sigma_1 \frac{k_1}{2} + \sigma_2 \frac{k_2}{2}\right) - \cdots$$

$$- (\tau_{m-1} - \tau_m)\left(v_b + \sigma_1 \frac{k_1}{2} + \cdots + \sigma_m \frac{k_m}{2}\right)$$

$$-\tau_m \left(v_b + \sigma_1 \frac{k_1}{2} + \cdots + \sigma_{m+1} \frac{k_{m+1}}{2}\right),$$

together with

$$x_c(0) = x_c - v_c(t - t_2) - (t_2 - \tau_1')\left(v_c - \sigma_1' \frac{k_1'}{2}\right) - (\tau_1' - \tau_2')$$

$$\times \left(v_c - \sigma_1' \frac{k_1'}{2} - \sigma_2' \frac{k_2'}{2}\right) - \cdots - (\tau_{p-1}' - \tau_p')\left(v_c - \sigma_1' \frac{k_1'}{2} - \cdots - \sigma_p' \frac{k_m'}{2}\right)$$

$$-\tau_p'\left(v_c - \sigma_1 \frac{k_1'}{2} - \cdots - \sigma_{p+1}' \frac{k_{p+1}'}{2}\right),$$

$$x_d(0) = x_d - (t_2 - \tau_1')\left(v_d + \sigma_1' \frac{k_1'}{2}\right) - (\tau_1' - \tau_2')\left(v_d + \sigma_1' \frac{k_1'}{2} + \sigma_2' \frac{k_2'}{2}\right) - \cdots$$

$$-(\tau'_{p-1}-\tau'_p)\left(v_d+\sigma'_1\frac{k'_1}{2}+\cdots+\sigma'_p\frac{k'_p}{2}\right)-\tau'_p$$
$$\times\left(v_d+\sigma'_1\frac{k'_1}{2}+\cdots+\sigma'_{p+1}\frac{k'_{p+1}}{2}\right).$$

Last, the initial velocities $v_a(0)$, $v_b(0)$, $v_c(0)$ and $v_d(0)$ satisfy

$$v_a(0)=v_a-\sigma_1\frac{k_1}{2}-\cdots-\sigma_{m+1}\frac{k_{m+1}}{2},$$
$$v_b(0)=v_b+\sigma_1\frac{k_1}{2}+\cdots+\sigma_{m+1}\frac{k_{m+1}}{2},$$

together with

$$v_c(0)=v_c-\sigma'_1\frac{k'_1}{2}-\cdots-\sigma'_{p+1}\frac{k'_{p+1}}{2},$$
$$v_d(0)=v_d+\sigma'_1\frac{k'_1}{2}+\cdots+\sigma'_{p+1}\frac{k'_{p+1}}{2}.$$

Changing variables as we did in Sec. 4, namely setting

$$t_1-\tau_1=\varepsilon s_1,\ldots,\tau_{m-1}-\tau_m=\varepsilon s_m,$$
$$t_2-\tau'_1=\varepsilon s'_1,\ldots,\tau'_{p-1}-\tau'_p=\varepsilon s'_m,$$
$$K_1=k_1,\ldots,K_m=k_m,\xi_b=(k_1+\cdots+k_{m+1})/\varepsilon,$$
$$K'_1=k'_1,\ldots,K'_m=k'_m,\xi_d=(k'_1+\cdots+k'_{p+1})/\varepsilon,$$
$$X_b=x_b-[x_a-v_a(t-t_1)],\qquad K_0=v_b-v_a,$$
$$X_d=x_d-[x_c-v_c(t-t_2)],\qquad K'_0=v_d-v_c,$$

gives eventually,

$$\left(S\,C^{m,\varepsilon}_{a,b}\,S\,C^{p,\varepsilon}_{c,d}\,S\,f_0^{\otimes 4}\right)(t,x_a,v_a,x_c,v_c)$$
$$=(-i)^{m+p+2}\sum_{\substack{\sigma_1,\ldots,\sigma_{m+1}\\ \sigma'_1,\ldots,\sigma'_{p+1}}}\sigma_1\cdots\sigma_{m+1}\sigma'_1\cdots\sigma'_{p+1}$$
$$\times\int_0^t dt_1\int_0^{t_1/\varepsilon}ds_1\cdots\int_0^{t_1/\varepsilon-s_1-\cdots-s_{m-1}}ds_m$$
$$\times\int_0^{t_1-\varepsilon(s_1+\cdots+s_m)}dt_2\int_0^{t_2/\varepsilon}ds'_1\cdots\int_0^{t_2/\varepsilon-s'_1-\cdots-s'_{p-1}}ds'_p$$
$$\times\int dX_b\,d\xi_b\,\frac{dK_0}{(2\pi)^3}\cdots\frac{dK_m}{(2\pi)^3}\,dX_d\,d\xi_d\,\frac{dK'_0}{(2\pi)^3}\cdots\frac{dK'_p}{(2\pi)^3}$$

$$\times \hat{\phi}(K_1)\cdots\hat{\phi}(K_m)\hat{\phi}(-(K_1+\cdots+K_m)+\varepsilon\xi_b)$$
$$\times \hat{\phi}(K_1')\cdots\hat{\phi}(K_m')\hat{\phi}(-(K_1'+\cdots+K_m')+\varepsilon\xi_d)$$
$$\times \exp\left(-iK\cdot QK - iK'\cdot Q'K'\right)$$
$$\times \exp\left(-iX_b\cdot\xi_b + i\varepsilon\xi_b\cdot MK - iX_d\cdot\xi_d + i\varepsilon\xi_d\cdot M'K'\right)$$
$$\times f_0(x_a(0), v_a(0))\ f_0(x_b(0), v_b(0))\ f_0(x_c(0), v_c(0))\ f_0(x_d(0), v_d(0)).$$

Here, the initial position and velocities have the value

$$x_a(0) = x_a - v_a t + [RKt_1 + \varepsilon NK + \varepsilon\mu\xi_b],$$
$$v_a(0) = v_a - [RK + \varepsilon\lambda\xi_b],$$
$$x_b(0) = x_a + X_b - v_a t - K_0 t_1 - [RKt_1 + \varepsilon NK + \varepsilon\mu\xi_b],$$
$$v_b(0) = v_a + K_0 + [RK + \varepsilon\lambda\xi_b],$$

together with,

$$x_c(0) = x_c - v_c t + [R'K't_2 + \varepsilon N'K' + \varepsilon\mu'\xi_d],$$
$$v_c(0) = v_c - [R'K' + \varepsilon\lambda'\xi_d],$$
$$x_d(0) = x_c + X_d - v_c t - K_0' t_2 - [R'K't_2 + \varepsilon N'K' + \varepsilon\mu'\xi_d],$$
$$v_d(0) = v_c + K_0' + [R'K' + \varepsilon\lambda'\xi_d].$$

Also, we used the notations

$$\begin{aligned}
K\cdot QK &= s_1K_1(K_0+\sigma_1K_1)+\cdots+s_m(K_1+\cdots+K_m)\\
&\quad\times(K_0+\sigma_1K_1+\cdots+\sigma_mK_m),\\
RK &= \frac{\sigma_1-\sigma_{m+1}}{2}K_1 + \frac{\sigma_2-\sigma_{m+1}}{2}K_2 + \cdots + \frac{\sigma_m-\sigma_{m+1}}{2}K_m,\\
\varepsilon MK &= \varepsilon s_1(K_0+\sigma_1K_1)+\varepsilon s_2(K_0+\sigma_1K_1+\sigma_2K_2)\\
&\quad+\cdots+\varepsilon s_m(K_0+\sigma_1K_1+\cdots+\sigma_mK_m),\\
\varepsilon l\xi_b &= \varepsilon\sigma_{m+1}\frac{\xi_b}{2},\\
\varepsilon NK &:= \varepsilon s_1(\sigma_1K_1)+\varepsilon s_2(\sigma_1K_1+\sigma_2K_2)+\cdots+\varepsilon s_m(\sigma_1K_1+\cdots+\sigma_mK_m)\\
&\quad-\varepsilon RK(s_1+\cdots+s_m),\\
\varepsilon\mu\xi_b &:= \varepsilon\lambda\xi_b(t_1-\varepsilon[s_1+\cdots+s_m]),
\end{aligned}$$

together with

$$K'\cdot Q'K' = s_1'K_1'(K_0'+\sigma_1'K_1')+\cdots+s_p'(K_1'+\cdots+K_p')$$

$$
\begin{aligned}
&\times(K_0' + \sigma_1' K_1' + \cdots + \sigma_p' K_p'),\\
R'K' &= \frac{\sigma_1' - \sigma_{p+1}'}{2} K_1' + \frac{\sigma_2' - \sigma_{p+1}'}{2} K_2' + \cdots + \frac{\sigma_p' - \sigma_{p+1}'}{2} K_p',\\
\varepsilon M'K' &= \varepsilon s_1'(K_0' + \sigma_1' K_1') + \varepsilon s_2'(K_0' + \sigma_1' K_1' + \sigma_2' K_2')\\
&\quad + \cdots + \varepsilon s_p'(K_0' + \sigma_1' K_1 + \cdots + \sigma_p' K_p'),\\
\varepsilon \lambda' \xi_d &= \varepsilon \sigma_{p+1}' \frac{\xi_d}{2},\\
\varepsilon N'K' &:= \varepsilon s_1'(\sigma_1' K_1') + \varepsilon s_2'(\sigma_1' K_1' + \sigma_2' K_2') + \cdots + \varepsilon s_p'(\sigma_1' K_1' + \cdots + \sigma_p' K_p')\\
&\quad - \varepsilon R'K'(s_1' + \cdots + s_p'),\\
\varepsilon \mu' \xi_d &:= \varepsilon \lambda' \xi_d (t_1 - \varepsilon[s_1' + \cdots + s_p']).
\end{aligned}
$$

From those formula, it is clear that the same approach than the one we developped in Sect. 4 above, exploiting here the independence of the variables (K, X_b, ξ_b) and (K', X_d, ξ_d), and the behaviour of the quadratic forms Q and Q', allows to bound

$$
\begin{aligned}
&\left\| \mathcal{F}\left(S\, C_{a,b}^{m,\varepsilon}\, S\, C_{c,d}^{p,\varepsilon}\, S\, f_0^{\otimes 4}\right)(t, \xi_a, \eta_a, \xi_c, \eta_c) \right\|_{L^1_{xia,\eta_a,\xi_c,\eta_c} \cap L^1_{xia,\xi_c} L^\infty_{\eta_a,\eta_c}}\\
&\qquad \le\ c_0^{m+p+2}\ N_0(\phi)^{m+p+2}\ N_1(f_0)^4 \int_0^t dt_1 \int_0^{t_1} dt_2,
\end{aligned}
$$

for the same universal constant c_0 as in Theorem 2. On the more, the limiting value $S\, C_{a,b}^{m,\varepsilon}\, S\, C_{c,d}^{p,\varepsilon}\, S\, f_0^{\otimes 4}$ clearly is

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \left(S\, C_{a,b}^{m,\varepsilon}\, S\, C_{c,d}^{p,\varepsilon}\, S\, f_0^{\otimes 4}\right)(t, x_a, v_a, x_c, v_c)\\
&= (-i)^{m+p+2} \sum_{\substack{\sigma_1, \ldots, \sigma_{m+1}\\ \sigma_1', \ldots, \sigma_{p+1}'}} \sigma_1 \cdots \sigma_{m+1} \sigma_1' \cdots \sigma_{p+1}'\\
&\quad \times \int_0^t dt_1 \int_0^{+\infty} ds_1 \cdots \int_0^{+\infty} ds_m \int_0^{t_1} dt_2 \int_0^{+\infty} ds_1' \cdots \int_0^{+\infty} ds_p'\\
&\quad \times \int dX_b\, d\xi_b\, \frac{dK_0}{(2\pi)^3} \cdots \frac{dK_m}{(2\pi)^3}\, dX_d\, d\xi_d\, \frac{dK_0'}{(2\pi)^3} \cdots \frac{dK_p'}{(2\pi)^3}\\
&\quad \times \hat{\phi}(K_1) \cdots \hat{\phi}(K_m) \hat{\phi}(-(K_1 + \cdots + K_m))\ \hat{\phi}(K_1') \cdots \hat{\phi}(K_m')\\
&\quad \times \hat{\phi}(-(K_1' + \cdots + K_m'))\\
&\quad \times \exp\left(-iK \cdot QK - iK' \cdot Q'K'\right)\ \exp\left(-iX_b \cdot \xi_b - iX_d \cdot \xi_d\right)\\
&\quad \times f_0(x_a - v_a t + RKt_1, v_a - RK)\ f_0(x_a + X_b - v_a t - K_0 t_1 - RKt_1, v_a
\end{aligned}
$$

$$+ K_0 + RK)\ f_0(x_c - v_c t + R'K't_2, v_c - R'K')\ f_0(x_c + X_d - v_c t - K_0' t_2 - R'K't_2, v_c + K_0' + R'K').$$

Performing the identification of the Born series expansion as we did in Sec.5.2, gives the

Theorem 5. *Let j be an integer. Let $r_1 \le j$ and $r_2 \le j+1$. Let c_0 be as in Theorem 2. The following convergence holds:*

$$\mathcal{F}_{x_a,v_a}\left(S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right)(t,\xi_a,\eta_a)$$
$$\longrightarrow \mathcal{F}_{x_a,v_a}\left(S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right)(t,\xi_a,\eta_a)$$
$$in\ C^0\left(\mathbb{R}^+_t; L^1\left(\mathbb{R}^6_{\xi_a,\eta_a}\right)\right)\ and\ in\ C^0\left(\mathbb{R}^+_t; L^1\left(\mathbb{R}^3_{\xi_a}; L^\infty\left(\mathbb{R}^3_{\eta_a}\right)\right)\right).$$

On the more, the following uniform bound holds true:

$$N_1\left[S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right] \le c_0^{m+p+2}$$
$$\times N_0(\phi)^{m+p+2}\ N_1(f_0)^{j+2}\int_0^t dt_1 \int_0^{t_1} dt_2.$$

7.2. Summing Over m and p: Analysis of $S\,C^{\varepsilon}_{r_1,j+1}\,S\,C^{\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}$

From Theorem 5 and the identity

$$S\,C^{\varepsilon}_{r_1,j+1}\,S\,C^{\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2} = \left(\sum_m S\,C^{\varepsilon}_{r_1,j+1}\right)\left(\sum_p S\,C^{\varepsilon}_{r_2,j+2}\right)S\,f_0^{\otimes j+2},$$

one immediately infers the

Theorem 6. *Let j be an integer. Let $r_1 \le j$ and $r_2 \le j+1$. Let c_0 be as in Theorem 2. Assume that the potential ϕ is small enough, in that*

$$N_0(\phi) \le c_1,$$

for the same universal constant $c_1 > 0$ as in Theorem 4. The following convergence holds:

$$\mathcal{F}_{x_a,v_a}\left(S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right)(t,\xi_a,\eta_a)$$
$$\longrightarrow \mathcal{F}_{x_a,v_a}\left(S\,C^{m,\varepsilon}_{r_1,j+1}\,S\,C^{p,\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right)(t,\xi_a,\eta_a)$$
$$in\ C^0\left(\mathbb{R}^+_t; L^1\left(\mathbb{R}^6_{\xi_a,\eta_a}\right)\right)\ and\ in\ C^0\left(\mathbb{R}^+_t; L^1\left(\mathbb{R}^3_{\xi_a}; L^\infty\left(\mathbb{R}^3_{\eta_a}\right)\right)\right).$$

On the more, the following uniform bound holds true:

$$N_1\left[S\,C^{\varepsilon}_{r_1,j+1}\,S\,C^{\varepsilon}_{r_2,j+2}\,S\,f_0^{\otimes j+2}\right]$$
$$\leq \left(\sum_{m\geq 1} c_0^{m+1}\ N_0(\phi)^{m+1}\right)^2\ N_1(f_0)^{j+2}\ \int_0^t dt_1 \int_0^{t_1} dt_2.$$

8. CONCLUSION: PROOF OF THE MAIN THEOREM

Linking n collisions as we did in Sec.7 for the case $n = 2$ allows to prove that the generic term

$$\left[\sum_{r_1=1}^{j} S\,C^{\varepsilon}_{r_1,j+1}\right]\left[\sum_{r_2=1}^{j+1} S\,C^{\varepsilon}_{r_2,j+2}\right]\cdots\left[\sum_{r_n=1}^{j+n-1} S\,C^{\varepsilon}_{r_n,j+n}\right] S\,f_0^{\otimes j+n},$$

in the expansion (2.33) goes, in the same topology as in Theorems 3, 4, 5, 6, towards

$$\left[\sum_{r_1=1}^{j} S\,C_{r_1,j+1}\right]\left[\sum_{r_2=1}^{j+1} S\,C_{r_2,j+2}\right]\cdots\left[\sum_{r_n=1}^{j+n-1} S\,C_{r_n,j+n}\right] S\,f_0^{\otimes j+n}.$$

In other words, the series expansion (2.33) of $\tilde{f}_j^N(t)$ converges term by term towards the series expansion (2.35) that defines $F_j(t)$. On the more, we readily have the uniform bound

$$N_1\left(\left[\sum_{r_1=1}^{j} S\,C^{\varepsilon}_{r_1,j+1}\right]\left[\sum_{r_2=1}^{j+1} S\,C^{\varepsilon}_{r_2,j+2}\right]\cdots\left[\sum_{r_n=1}^{j+n-1} S\,C^{\varepsilon}_{r_n,j+n}\right] S\,f_0^{\otimes j+n}\right)$$
$$\leq \left[\sum_{r_1=1}^{j}\left(\sum_{m\geq 1} c_0^{m+1}\ N_0(\phi)^{m+1}\right)\right]\cdots\left[\sum_{r_n=1}^{j+n-1}\left(\sum_{m\geq 1} c_0^{m+1}\ N_0(\phi)^{m+1}\right)\right]$$
$$\times \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$
$$\leq \left(\sum_{m\geq 1} c_0^{m+1}\ N_0(\phi)^{m+1}\right)^n\ \frac{j(j+1)\cdots(j+n-1)\times t^n}{n!},$$

a summable function of n (uniformly in ε), provided t is small enough. This achieves the proof of our main Theorem.

ACKNOWLEDGMENTS

This work has been partially supported by the European Program 'Improving the Human Potential' in the framework of the 'HYKE' network HPRN-CT-2002-00282, by the GDR "Amplitude Equations and Qualitative Properties" (GDR CNRS 2103 : EAPQ), and by the "ACI Jeunes Chercheurs—Méthodes haute fréquence pour les Equations différentielles ordinaires, et aux dérivées partielles. Applications."

REFERENCES

1. N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, (Saunders, Philadelphia, 1976).
2. D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti, Some considerations on the derivation of the nonlinear quantum boltzmann equation. *J. Stat. Phys.* **116**(1/4):381–410 (2004).
3. D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti, *On the Weak Coupling Limit for Bosons and Fermions*, to appear in *Math. Mod. Meth. Appl. Sci.* (2005).
4. A. Bohm, *Quantum Mechanics*, Texts and Monographs in Physics, Springer-Verlag, 1979).
5. C. Boldrighini, L. A. Bunimovich, and Ya. Sinai, On the Boltzmann equation for the Lorentz gas. *J. Stat. Phys.* **32**(3):477–501 (1983).
6. F. Castella, From the von Neumann equation to the Quantum Boltzmann equation in a deterministic framework. *J. Stat. Phys.* **104**(1/2):387–447 (2001).
7. F. Castella, From the von Neumann equation to the Quantum Boltzmann equation II: identifying the Born series. *J. Stat. Phys.* **106**(5/6):1197–1220 (2002).
8. F. Castella and A. Plagne, Non-derivation of the quantum Boltzmann equation from the periodic Schrödinger equation. *Indiana Univ. Math. J.* **51**(4):963–1016 (2002).
9. S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, (Cambridge Univ. Press, Cambridge, England, 1970).
10. C. Cercignani, R. Illner, and M. Pulvirenti, *The mathematical theory of dilute gases*, *Applied Mathematical Sciences* **106**, (Springer-Verlag, New York, 1994).
11. T. Chen, L^r *Convergence of a Random Schrödinger to a Linear Boltzmann Evolution*, preprint arxiv math-ph/0407037.
12. S. L. Chuang, *Physics of Optoelectronic Devices*, Wiley series in pure and applied optics, New-York (1995).
13. C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Mécanique Quantique*, I et II, Enseignement des Sciences **16**, Hermann (1973).
14. M. Combescot, On the generalized golden rule for transition probabilities. *Phys. A: Math. Gene.* **34**(31):6087–6104 (2001).
15. R. Dümcke, The low density limit for an N-level system interacting with a free Bose or Fermi gas. *Comm. Math. Phys.* **97**(3):331–359 (1985).
16. D. Dürr, S. Goldstein, and J. Lebowitz, Asymptotic motion of a classical particle in a random potential in two dimensions: the Landau model. *Comm. Math. Phys.* **113**:209–230 (1987).
17. D. Eng and L. Erdös, *The linear Boltzmann equation as the low density limit of a random Schrödinger equation*, preprint arxiv math-ph/0412044.
18. L. Erdös, M. Salmhofer, and H.-T. Yau, On the quantum Boltzmann equation, *Mathematical Physics Archive* University of Texas, 03–56, (2003).

19. L. Erdös and H. T. Yau, *Linear Boltzmann Equation as Scaling Limit of Quantum Lorentz Gas*, Advances in differential equations and mathematical physics (Atlanta, GA, 1997):137–155, *Contemp. Math.*, 217, *Amer. Math. Soc., Providence*, RI (1998).
20. L. Erdös and H. T. Yau, Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation, *Comm. Pure Appl. Math.* **53**(6):667–735 (2000).
21. R. Esposito, M. Pulvirenti, and A. Teta, The Boltzmann Equation for a One-Dimensional Quantum Lorentz gas, *Comm. Math. Phys.* **204**(3):619–649 (1999), and *Erratum:* "The Boltzmann equation for a one-dimensional quantum Lorentz gas," *Comm. Math. Phys.* **214**(2):493–494 (2000).
22. M. V. Fischetti, Theory of electron transport in small semiconductor devices using the Pauli master equations. *J. Appl. Phys.* **83**(1):270–291 (1998).
23. G. Gallavotti, *Time evolution problems in classical statistical mechanics and the wind-tree-model Cargese* Lectures in Physics, Vol. IV, ed. D. Kastler, Gordon Breach, Paris, (1970); Divergences and approach to equilibrium in the Lorenz and the wind-tree models. *Phys. Rev.* **185**:308–322 (1969). See also the book *Statistical Mechanics*, Appendix A2 to Ch. 1, Springer–Verlag (1999).
24. T. G. Ho and L. J. Landau, Fermi gas on a lattice in the van Hove limit. *J. Stat. Phys.* **87**(3–4):821–845 (1997).
25. T. G. Ho, L. J. Landau, and A. J. Wilkins, On the weak coupling limit for a Fermi gas in a random potential. *Rev. Math. Phys.* **5**(2):209–298 (1993).
26. N. M. Hugenholtz, Derivation of the Boltzmann equation for a Fermi gas. *J. Stat. Phys.* **32**(2):231–254 (1983).
27. R. Illner and M. Pulvirenti, Global Validity of the Boltzmann equation for a two dimensional rare gas in vacuum. *Comm. Math. Phys.* **105**:189–203 (1986). Erratum and improved result. *Comm. Math. Phys.* **121**:143–146 (1989).
28. J. B. Keller, G. Papanicolaou, and L. Ryzhik, Transport equations for elastic and other waves in random media. *Wave Motion* **24**(4):327–370 (1996).
29. W. Kohn and J. M. Luttinger. *Phys. Rev.* **108**:590 (1957).
30. W. Kohn and J. M. Luttinger. *Phys. Rev.* **109**:1892 (1958).
31. R. Kubo. *J. Phys. Soc. Jap.* **12**:570 (1957).
32. O. Lanford III, Time evolution of large classical systems. *Lecture Notes in Physics* **38**:1–111, E. J. Moser ed., Springer-Verlag (1975).
33. L. J. Landau, Observation of Quantum Particles on a Large Space-Time Scale. *J. Stat. Phys.* **77**(1–2):259–309 (1994).
34. P. A. Markowich, C. Ringhofer, and C. Schmeiser, *Semiconductor equations*, (Springer-Verlag, Vienna, 1990).
35. F. Nier, Asymptotic Analysis of a scaled Wigner equation and Quantum Scattering. *Transp. Theor. Stat. Phys.* **24**(4 et 5):591–629 (1995).
36. F. Nier, A semi-classical picture of quantum scattering. *Ann. Sci. Ec. Norm. Sup.* **4**(Sér., t. 29):149–183 (1996).
37. F. Poupaud and A. Vasseur, Classical and quantum transport in random media. *J. Mathématiques Pures et Appliquées* **82**(6):711–748 (2003).
38. W. Pauli, *Festschrift zum 60. Geburtstage A. Sommerfelds*, p. 30, (Hirzel, Leipzig, 1928).
39. M. Reed and B. Simon, *Methods of modern mathematical physics III. Scattering theory* (Academic Press, New York-London, 1979).
40. E. Rosencher and B. Vinter, *Optoelectronique*, Dunod (2002).
41. H. Spohn, Derivation of the transport equation for electrons moving through random impurities. *J. Stat. Phys.* **17**(6):385–412 (1977).
42. H. Spohn, *Large Scale Dynamics of Interacting Particles*, Springer (1991).
43. H. Spohn, Kinetic equations from Hamiltonian dynamics: Markovian limits, *Rev. Modern Phys.* **52**(3):569–615 (1980).

44. H. Spohn, Quantum Kinetic Equations, in *On Three Levels: Micro-Meso and Macro Approaches in Physics*, M. Fannes, C. Maes, A. Verbeure eds, Nato ASI Series B: *Physics,* V. **324**: 1–10 (1994).
45. L. Van Hove. *Physica* **21**:517 (1955).
46. L. Van Hove. *Physica* **23**:441 (1957).
47. L. Van Hove, in *Fundamental Problems in Statistical Mechanics*, E.G.D. Cohen ed., 157 (1962).
48. R. Zwanzig, *Quantum Statistical Mechanics*, P.H.E. Meijer ed., (Gordon and Breach, New-York, 1966).